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ON HARDY-HILBERT INTEGRAL INEQUALITIES WITH SOME PARAMETERS

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Abstract

In this paper, we give a new Hardy-Hilbert's integral inequality with some parameters and a best constant factor. It includes an overwhelming majority of results of many papers.

2000 Mathematics Subject Classification: 26D15.

Key words: Hardy-Hilbert's integral inequality, Weight, Parameter, Best constant factor, β -function, Γ -function.

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1. Introduction and Main Result

Let $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $f \geq 0$, $g \geq 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then we have the well known Hardy-Hilbert inequality

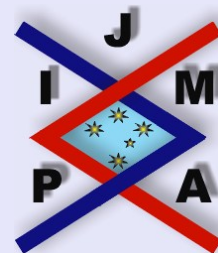
$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}};$$

and an equivalent form as:

$$(1.2) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)dx.$$

In recent years, many results have been obtained in the research of these two inequalities (see [1] – [13]). Yang [1] and [2] gave:

$$(1.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{p+\lambda-2}{p}, \frac{p+\lambda-2}{q}\right) \times \left(\int_0^\infty x^{1-\lambda} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda} g^q(x)dx\right)^{\frac{1}{q}},$$



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where $B(r, s)$ is the β -function; and Kuang [3] gave:

$$(1.4) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin^{\frac{1}{p}}\left(\frac{\pi}{p\lambda}\right) \sin^{\frac{1}{q}}\left(\frac{\pi}{q\lambda}\right)} \left(\int_0^\infty x^{1-\lambda} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda} g^q(x) dx\right)^{\frac{1}{q}}.$$

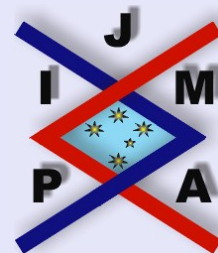
Recently, Hong [4] gave:

$$(1.5) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{\sqrt{x^2 + y^2}} dx dy \leq \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2q}\right) \left(\int_0^\infty f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x) dx\right)^{\frac{1}{q}}.$$

And Yang [5] gave:

$$(1.6) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{(q-1)(1-\lambda)} g^q(x) dx\right)^{\frac{1}{q}};$$

$$(1.7) \quad \int_0^\infty y^{\lambda-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx\right)^p dy$$



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$$< \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx.$$

These results generalize and improve (1.1) and (1.2) in a certain degree.

In this paper, by introducing a few parameters, we obtain a new Hardy-Hilbert integral inequality with a best constant factor, which is a more extended inequality, and includes all the results above and the overwhelming majority of results of many recent papers.

Our main result is as follows:

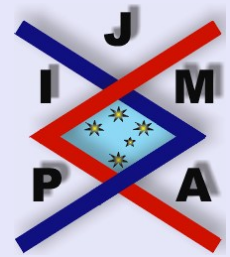
Theorem 1.1. *If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\alpha > 0$, $\lambda > 0$, $m, n \in \mathbb{R}$, such that $0 < 1 - mp < \alpha\lambda$, $0 < 1 - nq < \alpha\lambda$, and $f \geq 0$, $g \geq 0$, satisfy*

$$(1.8) \quad 0 < \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx < \infty,$$

$$(1.9) \quad 0 < \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y) dy < \infty,$$

then

$$(1.10) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy < H_{\lambda,\alpha}(m, n, p, q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx \right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y) dy \right)^{\frac{1}{q}};$$



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and

$$(1.11) \quad \int_0^\infty y^{\frac{(1-\alpha\lambda)+q(m-n)}{1-q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^p dy < \tilde{H}_{\lambda,\alpha}(m, n, p, q) \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx,$$

where

$$H_{\lambda,\alpha}(m, n, p, q) = \frac{1}{\alpha} B^{\frac{1}{p}} \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) B^{\frac{1}{q}} \left(\frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha} \right)$$

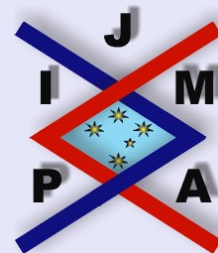
and

$$\tilde{H}_{\lambda,\alpha}(m, n, p, q) = \frac{1}{\alpha^p} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) B^{p-1} \left(\frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha} \right).$$

Theorem 1.2. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\lambda > 0$, $m, n \in \mathbb{R}$, such that $0 < 1 - mp < \alpha\lambda$, $mp + nq = 2 - \alpha\lambda$, and $f(x) \geq 0$, $g(y) \geq 0$, satisfy

$$(1.12) \quad 0 < \int_0^\infty x^{n(p+q)-1} f^p(x) dx < \infty,$$

$$(1.13) \quad 0 < \int_0^\infty y^{m(p+q)-1} g^q(y) dy < \infty,$$



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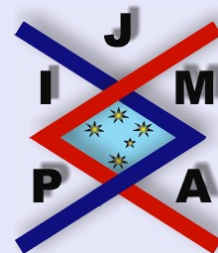
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then

$$(1.14) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy < \frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) \\ \times \left(\int_0^\infty x^{n(p+q)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{m(p+q)-1} g^q(y) dy \right)^{\frac{1}{q}};$$

$$(1.15) \quad \int_0^\infty y^{\frac{m(p+q)-1}{1-q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^p dy \\ < \frac{1}{\alpha^p} B^p \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) \int_0^\infty x^{n(p+q)-1} f^p(x) dx,$$

where the constant factors $\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ in (1.14) and $\frac{1}{\alpha^p} B^p \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ in (1.15) are the best possible.



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2. Weight Function and Lemmas

The weight function is defined as follows

$$\omega_{\lambda,\alpha}(m, n, y) = \int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} \cdot \frac{y^n}{x^m} dx, \quad y \in (0, +\infty).$$

Lemma 2.1. *If $\alpha > 0$, $\lambda > 0$, $m \in \mathbb{R}$, $0 < 1 - m < \alpha\lambda$, then*

$$(2.1) \quad \omega_{\lambda,\alpha}(m, n, y) = \frac{1}{\alpha} y^{(1-\alpha\lambda)+(n-m)} B\left(\frac{1-m}{\alpha}, \lambda - \frac{1-m}{\alpha}\right).$$

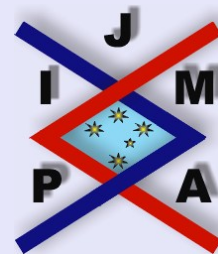
Proof. Setting $t = \frac{x^\alpha}{y^\alpha}$, then

$$\begin{aligned} \omega_{\lambda,\alpha}(m, n, y) &= \frac{1}{\alpha} \int_0^\infty \frac{1}{(1+t)^\lambda} y^{(1-\alpha\lambda)+(n-m)} t^{\frac{1-m}{\alpha}-1} dt \\ &= \frac{1}{\alpha} y^{(1-\alpha\lambda)+(n-m)} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-m}{\alpha}-1} dt \\ &= \frac{1}{\alpha} y^{(1-\alpha\lambda)+(n-m)} B\left(\frac{1-m}{\alpha}, \lambda - \frac{1-m}{\alpha}\right). \end{aligned}$$

Hence (2.1) is valid. The lemma is proved. □

Lemma 2.2. *If $\alpha > 0$, $\lambda > 0$, $\beta < 1$, $a \in \mathbb{R}$, then*

$$(2.2) \quad \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-\beta}{\alpha}-1-a\varepsilon} dt dx = O(1), \quad (\varepsilon \rightarrow 0^+).$$



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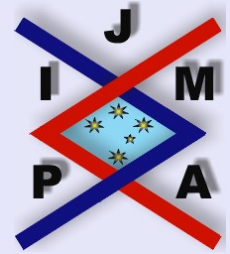
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Proof. Since $(1 - \beta)/\alpha > 0$, for ε small enough, such that $\frac{1-\beta}{\alpha} - a\varepsilon > 0$, then

$$\begin{aligned} \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-\beta}{\alpha}-1-a\varepsilon} dt dx &< \int_1^\infty \frac{1}{x} \int_0^{\frac{1}{x^\alpha}} t^{(\frac{1-\beta}{\alpha}-a\varepsilon)-1} dt dx \\ &= \frac{1}{1-\beta-a\varepsilon\alpha} \int_1^\infty x^{\beta+a\varepsilon\alpha-2} dx \\ &= \frac{1}{(1-\beta-a\varepsilon\alpha)^2}. \end{aligned}$$

Hence (2.2) is valid. □



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3. Proofs of the Theorems

Proof of Theorem 1.1. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
 G &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy \\
 &= \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x^\alpha + y^\alpha)^{\lambda/p}} \frac{x^n}{y^m} \right] \left[\frac{g(y)}{(x^\alpha + y^\alpha)^{\lambda/q}} \frac{y^m}{x^n} \right] dx dy \\
 (3.1) \quad &\leq \left\{ \int_0^\infty \int_0^\infty \frac{f^p(x)}{(x^\alpha + y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}} \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \frac{g^q(y)}{(x^\alpha + y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}} dx dy \right\}^{\frac{1}{q}},
 \end{aligned}$$

according to the condition of taking equality in Hölder's inequality, if (3.1) takes equality, then there exists a constant C , such that

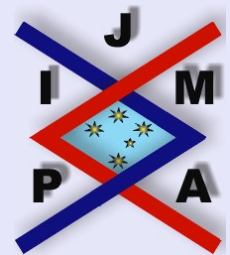
$$\left[\frac{f^p(x)}{(x^\alpha + y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}} \right] / \left[\frac{g^q(y)}{(x^\alpha + y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}} \right] \equiv C, \text{ a.e. } (x, y) \in (0, +\infty) \times (0, +\infty)$$

it follows that

$$f^p(x)x^{n(p+q)} \equiv Cg^q(y)y^{m(p+q)} \equiv C_1 \text{ (constant), a.e. } (x, y) \in (0, +\infty) \times (0, +\infty)$$

hence

$$\begin{aligned}
 &\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx \\
 &= \int_0^\infty x^{(1-\alpha\lambda)+n(p+q)-nq-mp} f^p(x) dx \\
 &= C_1 \int_0^\infty x^{(1-\alpha\lambda)-np-mq} dx
 \end{aligned}$$



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$$= C_1 \int_0^1 x^{(1-\alpha\lambda)-np-mq} dx + C_1 \int_1^\infty x^{(1-\alpha\lambda)-np-mq} dx = +\infty,$$

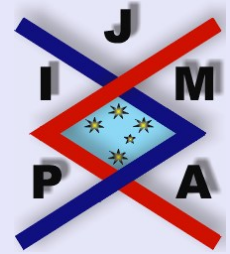
which contradicts (1.8) and (1.9). Hence, by (3.1), we have

$$\begin{aligned} G &< \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}} dy \right] g^q(y) dy \right\}^{\frac{1}{q}} \\ &= \left\{ \frac{1}{\alpha} \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \frac{1}{\alpha} \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} B\left(\frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha}\right) g^q(y) dy \right\}^{\frac{1}{q}} \\ &= H_{\lambda,\alpha}(m, n, p, q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y) dy \right)^{\frac{1}{q}}. \end{aligned}$$

Hence (1.10) is valid.

Let $\beta = [(1 - \alpha\lambda) + q(m - n)]/(1 - q)$, and

$$\tilde{g}(y) = y^\beta \left(\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right)^{\frac{p}{q}}.$$



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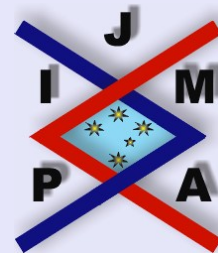
By (1.10), we have

$$\begin{aligned}
 & \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} \tilde{g}^q(y) dy \\
 &= \int_0^\infty y^{\beta(1-q)} \tilde{g}^q(y) dy \\
 &= \int_0^\infty y^\beta \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^p dy \\
 &= \int_0^\infty y^\beta \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^{\frac{p}{q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right] dy \\
 &= \int_0^\infty \int_0^\infty \frac{f(x) \tilde{g}(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy \\
 &< H_{\lambda,\alpha}(m, n, p, q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx \right)^{\frac{1}{p}} \\
 &\quad \times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} \tilde{g}^q(y) dy \right)^{\frac{1}{q}},
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & \int_0^\infty y^{\frac{(1-\alpha\lambda)+q(m-n)}{1-q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^p dy \\
 &\quad < \tilde{H}_{\lambda,\alpha}(m, n, p, q) \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x) dx.
 \end{aligned}$$

Hence, (1.11) is valid. \square



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Proof of Theorem 1.2. Since $0 < 1 - mp < \alpha\lambda$, $mp + nq = 2 - \alpha\lambda$, then

$$\begin{aligned} 0 < 1 - nq < \alpha\lambda, \\ (1 - \alpha\lambda) + p(n - m) &= n(p + q) - 1, \\ (1 - \alpha\lambda) + q(m - n) &= m(p + q) - 1, \\ \frac{1 - mp}{\alpha} &= \lambda - \frac{1 - nq}{\alpha}. \end{aligned}$$

By Theorem 1.1, (1.14) and (1.15) are valid.

For $\varepsilon > 0$, setting

$$f_0(x) = \begin{cases} x^{[-n(p+q)-\varepsilon]/p}, & x \geq 1; \\ 0, & 0 \leq x < 1, \end{cases}$$

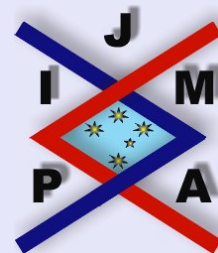
and

$$g_0(y) = \begin{cases} y^{[-m(p+q)-\varepsilon]/q}, & y \geq 1; \\ 0, & 0 \leq y < 1. \end{cases}$$

We have

$$(3.2) \quad 0 < \int_0^\infty x^{n(p+q)-1} f_0^p(x) dx = \int_1^\infty x^{-1-\varepsilon} dx = \frac{1}{\varepsilon} < \infty,$$

$$(3.3) \quad 0 < \int_0^\infty y^{m(p+q)-1} g_0^q(y) dy = \int_1^\infty y^{-1-\varepsilon} dy = \frac{1}{\varepsilon} < \infty.$$



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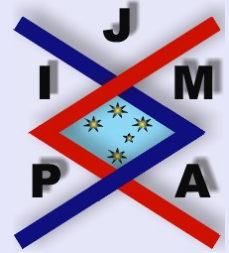


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$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy \\
 &= \int_1^\infty \int_1^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} x^{-\frac{n(p+q)+\varepsilon}{p}} y^{-\frac{m(p+q)+\varepsilon}{q}} dx dy \\
 &= \int_1^\infty x^{-\frac{n(p+q)+\varepsilon}{p}} \int_1^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} y^{-\frac{m(p+q)+\varepsilon}{q}} dy dx \\
 &= \frac{1}{\alpha} \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_{\frac{1}{x^\alpha}}^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt dx \\
 &= \frac{1}{\alpha} \left[\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt dx \right. \\
 &\quad \left. - \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt dx \right].
 \end{aligned}$$

By Lemma 2.2, when $\varepsilon \rightarrow 0$, we have

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt dx = O(1).$$

Since

$$\int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt = B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) + o(1),$$

we have

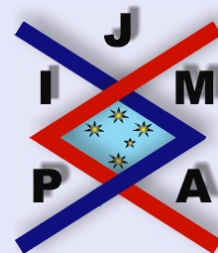
$$\begin{aligned}
 \int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy &= \frac{1}{\alpha} \left\{ \frac{1}{\varepsilon} \left[B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) + o(1) \right] - O(1) \right\} \\
 &= \frac{1}{\varepsilon} \left[\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) - o(1) \right] \\
 (3.4) \qquad \qquad \qquad &= \frac{1}{\varepsilon \alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) (1 - o(1)).
 \end{aligned}$$

If the constant $\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ in (1.14) is not the best possible, then there exists a $K < \frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$, such that (1.14) still is valid when we replace $\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ by K . By (3.2), (3.3) and (3.4), we find

$$\begin{aligned}
 &\frac{1}{\varepsilon \alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) (1 - o(1)) \\
 &< K \left(\int_0^\infty x^{n(p+q)-1} f_0^n(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{m(p+q)-1} g_0^q(y) dy \right)^{\frac{1}{q}} = K \frac{1}{\varepsilon}.
 \end{aligned}$$

For $\varepsilon \rightarrow 0^+$, we have $\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) \leq K$, which contradicts the fact that $K < \frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$. It follows that $\frac{1}{\alpha} B \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ in (1.14) is the best possible.

Since (1.14) is equivalent to (1.15), then the constant $\frac{1}{\alpha^p} B^p \left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right)$ in (1.15) is the best possible. The theorem is proved. \square



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4. Some Corollaries

When we take the appropriate parameters, many new inequalities can be obtained as follows:

Corollary 4.1. *If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\alpha > 0$, $\lambda > 0$, $f \geq 0$, $g \geq 0$, and $x^{(1-\alpha\lambda)(p-1)/p} f(x) \in L^p(0, +\infty)$, $x^{(1-\alpha\lambda)(q-1)/q} g(x) \in L^q(0, +\infty)$, then*

$$(4.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy$$

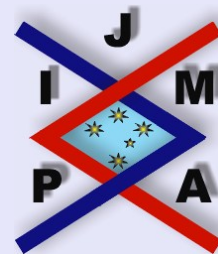
$$< \frac{\Gamma\left(\frac{\lambda}{p}\right) \Gamma\left(\frac{\lambda}{q}\right)}{\alpha\Gamma(\lambda)} \left(\int_0^\infty x^{(1-\alpha\lambda)(p-1)} f^p(x) dx \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty x^{(1-\alpha\lambda)(q-1)} g^q(x) dx \right)^{\frac{1}{q}};$$

$$(4.2) \quad \int_0^\infty y^{\alpha-1} \left(\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right)^p dy$$

$$< \left[\frac{\Gamma\left(\frac{\lambda}{p}\right) \Gamma\left(\frac{\lambda}{q}\right)}{\alpha\Gamma(\lambda)} \right]^p \int_0^\infty x^{(1-\alpha\lambda)(p-1)} f^p(x) dx,$$

where the constants $\Gamma\left(\frac{\lambda}{p}\right) \Gamma\left(\frac{\lambda}{q}\right) / (\alpha\Gamma(\lambda))$ in (4.1) and $\left[\Gamma\left(\frac{\lambda}{p}\right) \Gamma\left(\frac{\lambda}{q}\right) / (\alpha\Gamma(\lambda)) \right]^p$ in (4.2) are the best possible.



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Proof. If we take $m = \frac{1}{p} - \frac{\alpha\lambda}{p^2}$, $n = \frac{1}{q} - \frac{\alpha\lambda}{q^2}$ in Theorem 1.2, (4.1) and (4.2) can be obtained. \square

Corollary 4.2. *If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\lambda > 0$, $f \geq 0$, $g \geq 0$ and $x^{(1-\lambda)(p-1)/p} f(x) \in L^p(0, +\infty)$, $x^{(1-\lambda)(q-1)/q} g(x) \in L^q(0, +\infty)$, then*

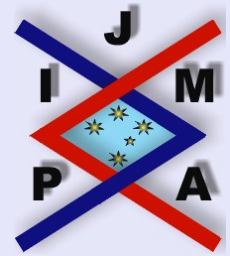
$$(4.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < \frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)} \left(\int_0^\infty x^{(1-\lambda)(p-1)} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{(1-\lambda)(q-1)} g^q(x) dx\right)^{\frac{1}{q}},$$

$$(4.4) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx\right)^p dy < \left[\frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}\right]^p \int_0^\infty x^{(1-\lambda)(p-1)} f^p(x) dx,$$

where $\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)/\Gamma(\lambda)$ in (4.3) and $\left[\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)/\Gamma(\lambda)\right]^p$ in (4.4) are the best possible.

Proof. If we take $\alpha = 1$ in Corollary 4.1, (4.3) and (4.4) can be obtained. \square

Corollary 4.3. *If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\lambda > 0$, $p + \lambda - 2 > 0$, $q + \lambda - 2 > 0$,*



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$f \geq 0, g \geq 0$, and $x^{(1-\lambda)/p}f(x) \in L^p(0, +\infty)$, $x^{(1-\lambda)/q}g(x) \in L^q(0, +\infty)$, then

$$(4.5) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy$$

$$< B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \left(\int_0^\infty x^{1-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda} g^q(x) dx \right)^{\frac{1}{q}},$$

$$(4.6) \quad \int_0^\infty y^{\frac{1-\lambda}{1-q}} \left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right)^p dy$$

$$< B^p \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \int_0^\infty x^{1-\lambda} f^p(x) dx,$$

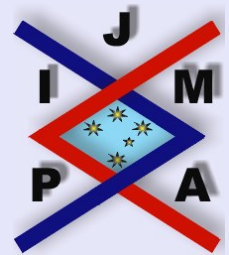
where $B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right)$ in (4.5) and $B^p \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right)$ in (4.6) are the best possible.

Proof. If we take $\alpha = 1, m = n = \frac{2-\lambda}{pq}$ in Theorem 1.2, (4.5) and (4.6) can be obtained. \square

Corollary 4.4. If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\alpha > 0, f \geq 0, g \geq 0$, and $x^{(1-\alpha)/p}f(x) \in L^p(0, +\infty)$, $x^{(1-\alpha)/q}g(x) \in L^q(0, +\infty)$, then

$$(4.7) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^\alpha + y^\alpha} dx dy$$

$$< \frac{\pi}{\alpha \sin^{\frac{1}{p}} \left(\frac{\pi}{p\alpha} \right) \sin^{\frac{1}{q}} \left(\frac{\pi}{q\alpha} \right)} \left(\int_0^\infty x^{1-\alpha} f^p(x) dx \right)^p \left(\int_0^\infty x^{1-\alpha} g^q(x) dx \right)^q,$$



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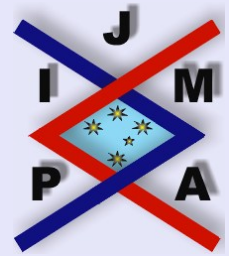


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$$(4.8) \quad \int_0^\infty y^{\frac{1-\alpha}{1-q}} \left(\int_0^\infty \frac{f(x)}{x^\alpha + y^\alpha} dx \right)^p dy < \left(\frac{\pi}{\alpha \sin^{\frac{1}{p}} \left(\frac{\pi}{p\alpha} \right) \sin^{\frac{1}{q}} \left(\frac{\pi}{q\alpha} \right)} \right)^p \int_0^\infty x^{1-\alpha} f^p(x) dx.$$

Proof. If we take $\lambda = 1$, $m = n = \frac{1}{pq}$, in Theorem 1.1, (4.7) and (4.8) can be obtained. \square

Corollary 4.5. *If $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $\alpha > 0$, $f \geq 0$, $g \geq 0$, and $f(x) \in L^p(0, +\infty)$, $g(x) \in L^q(0, +\infty)$, then*

$$(4.9) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{(x^\alpha + y^\alpha)^{\frac{1}{\alpha}}} dx dy < \frac{\Gamma\left(\frac{1}{p\alpha}\right) \Gamma\left(\frac{1}{q\alpha}\right)}{\alpha \Gamma\left(\frac{1}{\alpha}\right)} \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x) dx \right)^{\frac{1}{q}},$$

$$(4.10) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^{\frac{1}{\alpha}}} dx \right)^p dy < \left(\frac{\Gamma\left(\frac{1}{p\alpha}\right) \Gamma\left(\frac{1}{q\alpha}\right)}{\alpha \Gamma\left(\frac{1}{\alpha}\right)} \right)^p \int_0^\infty f^p(x) dx,$$

where $\Gamma\left(\frac{1}{p\alpha}\right) \Gamma\left(\frac{1}{q\alpha}\right) / (\alpha \Gamma\left(\frac{1}{\alpha}\right))$ in (4.9) and $\left[\Gamma\left(\frac{1}{p\alpha}\right) \Gamma\left(\frac{1}{q\alpha}\right) / (\alpha \Gamma\left(\frac{1}{\alpha}\right)) \right]^p$ in (4.10) are the best possible.

Proof. If we take $\lambda = \frac{1}{\alpha}$, $m = n = \frac{1}{pq}$ in Theorem 1.2, (4.9) and (4.10) can be obtained. \square

Remark 1. (4.1) and (4.2) are respectively generalizations of (1.6) and (1.7). For $\alpha = 1$ in (4.1) and (4.2), (1.6) and (1.7) can be obtained.

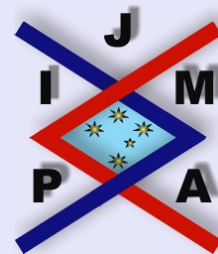
Remark 2. (4.3) and (4.4) are respectively generalizations of (1.1) and (1.2).

Remark 3. (4.5) is the result of [1] and [2]. (4.6) is a new inequality.

Remark 4. (4.7) is the result of [3]. (1.7) is a new inequality.

Remark 5. (4.9) is a generalization of (1.5). (4.10) is a new inequality.

For other appropriate values of parameters taken in Theorems 1.1 and 1.2, many new inequalities and the inequalities of [6] – [13] can yet be obtained.



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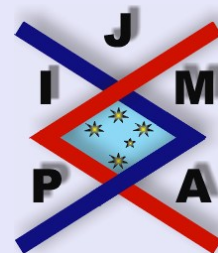
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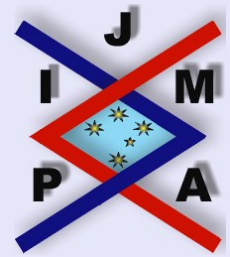
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