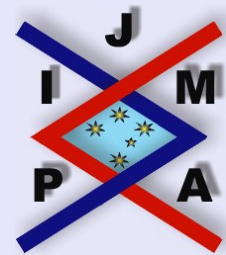


Journal of Inequalities in Pure and Applied Mathematics

NOTE ON THE CARLEMAN'S INEQUALITY FOR A NEGATIVE POWER NUMBER

THANH LONG NGUYEN, VU DUY LINH NGUYEN AND
THI THU VAN NGUYEN

Department of Mathematics and Computer Science,
College of Natural Science,
Vietnam National University HoChiMinh City,
227 Nguyen Van Cu Str.,
Dist.5, HoChiMinh City, Vietnam.
EMail: longnt@hcmc.netnam.vn



volume 4, issue 1, article 2,
2003.

*Received 30 October, 2002;
accepted 25 November, 2002.*

Communicated by: H. Bor

Abstract

Contents

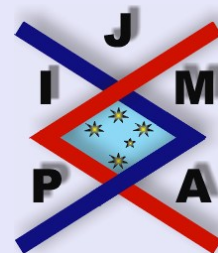


Home Page

Go Back

Close

Quit



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Abstract

By the method of indeterminate coefficients we prove the inequality

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n,$$

where $a_n > 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$.

2000 Mathematics Subject Classification: 26D15.

Key words: Carleman's inequality.

Contents

1	Introduction	3
2	Main Result	6
	References	

Title Page

Contents



Go Back

Close

Quit

Page 2 of 14

1. Introduction

The following Carleman inequality is well known (see [1, Chapter 9.12]).

$$(1.1) \quad \sum_{n=1}^{\infty} \left(\frac{a_1^{\frac{1}{p}} + a_2^{\frac{1}{p}} + \cdots + a_n^{\frac{1}{p}}}{n} \right)^p \leq \left(\frac{p}{p-1} \right) \sum_{n=1}^{\infty} a_n,$$

where $a_n \geq 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$, and $p > 1$.

Letting $p \rightarrow +\infty$, it follows from (1.1) that

$$(1.2) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n.$$

In practice, the inequality (1.2) is strict; i.e.,

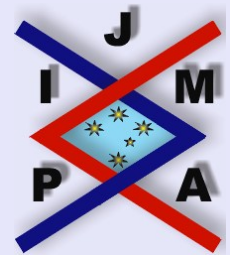
$$(1.3) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n,$$

if $a_n \geq 0$, $n = 1, 2, \dots$, $0 < \sum_{n=1}^{\infty} a_n < \infty$.

The constant e is sharp in the sense that it cannot be replaced by a smaller one.

Recently, the inequality (1.3) has also been improved by many authors, for example: Yang Bicheng and L. Debnath [2] with

$$(1.4) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n+2} \right) a_n,$$



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 3 of 14

in [3] by Yan Ping and Sun Guozheng with

$$(1.5) \quad \sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n + 1/5}\right)^{-1/2} a_n,$$

and in [4] by X. Yang with

$$(1.6) \quad \sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)} - \frac{1}{24(n+1)^2} - \frac{1}{48(n+1)^3}\right) a_n,$$

and

$$(1.7) \quad \sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^6 \frac{b_k}{(n+1)^k}\right) a_n,$$

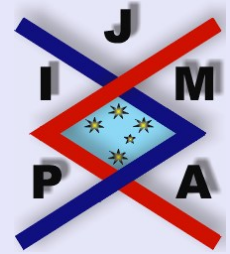
where

$$b_1 = \frac{1}{2}, b_2 = \frac{1}{24}, b_3 = \frac{1}{48}, b_4 = \frac{73}{5760}, b_5 = \frac{11}{1280}, b_6 = \frac{1945}{580608},$$

$$a_n \geq 0, n = 1, 2, \dots, 0 < \sum_{n=1}^{\infty} a_n < \infty.$$

We rewrite the inequality (1.1) with $r = \frac{1}{p}$ as follows

$$(1.8) \quad \sum_{n=1}^{\infty} \left(\frac{a_1^r + a_2^r + \dots + a_n^r}{n}\right)^{1/r} \leq (1-r)^{-1/r} \sum_{n=1}^{\infty} a_n,$$



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 4 of 14

where $a_n \geq 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$ and $0 < r < 1$.

In [5], we have improved Carleman's inequality (1.8) for a negative power number $r < 0$ as follows

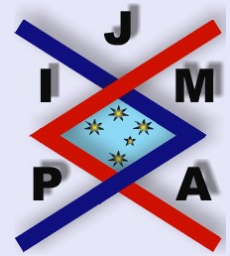
$$(1.9) \quad \sum_{n=1}^{\infty} \left(\frac{a_1^r + a_2^r + \dots + a_n^r}{n} \right)^{1/r} \leq \begin{cases} (1-r)^{-1/r} \sum_{n=1}^{\infty} a_n, & \text{if } -1 \leq r < 1, r \neq 0, \\ \frac{r}{r-1} 2^{(r-1)/r} \sum_{n=1}^{\infty} a_n, & \text{if } r < -1, \end{cases}$$

where $a_n > 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$.

In the case of $r = -1$, we obtain from (1.9) the inequality

$$(1.10) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} a_n,$$

where $a_n > 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$.



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 5 of 14

2. Main Result

In this paper, we shall prove the following theorem.

Theorem 2.1. *Let $a_n > 0$, $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} a_n < \infty$. Then we have*

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n.$$

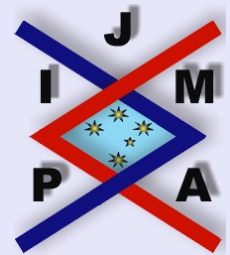
Remark 2.1. *From the inequality (2.1), we obtain the following inequalities:*

$$(2.2) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right) a_n,$$

$$(2.3) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} \right) a_n,$$

and

$$(2.4) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} a_n.$$



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

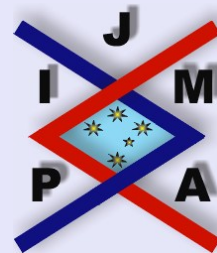
Page 6 of 14

Indeed, we note that the inequalities (2.2), (2.3), (2.4) are implied from (2.1), because

$$\begin{aligned}
 (2.5) \quad 1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \\
 < 1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \\
 < 1 - \frac{1}{3n+1} < 1.
 \end{aligned}$$

Hence, we obtain from (2.1), (2.5) that

$$\begin{aligned}
 (2.6) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \\
 \leq 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n \\
 = 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right. \\
 \quad \left. - 4n^2 \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n \\
 = 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right) a_n
 \end{aligned}$$



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents

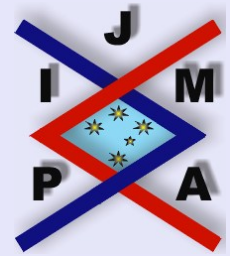


Go Back

Close

Quit

Page 7 of 14



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

$$\begin{aligned}
 & -8 \sum_{n=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) n^2 a_n \\
 & < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right) a_n \\
 & < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} \right) a_n \\
 & < 2 \sum_{n=1}^{\infty} a_n.
 \end{aligned}$$

To prove Theorem 2.1, we first prove the following lemma.

Lemma 2.2. *We have*

$$(2.7) \quad \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq \frac{a_1 b_1^2 + a_2 b_2^2 + \dots + a_n b_n^2}{(b_1 + b_2 + \dots + b_n)^2},$$

where $a_k > 0$, $b_k > 0$, $\forall k = 1, 2, \dots, n$.

Proof. This is a simple application of the Cauchy-Schwartz inequality

$$(2.8) \quad (b_1 + b_2 + \dots + b_n)^2 \leq \left(\frac{1}{\sqrt{a_1}} \sqrt{a_1} b_1 + \frac{1}{\sqrt{a_2}} \sqrt{a_2} b_2 + \dots + \frac{1}{\sqrt{a_n}} \sqrt{a_n} b_n \right)^2$$

Title Page

Contents



Go Back

Close

Quit

Page 8 of 14

$$\leq \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) (a_1 b_1^2 + a_2 b_2^2 + \cdots + a_n b_n^2).$$

□

Proof of Theorem 2.1. We prove the theorem by the method of indeterminate coefficients.

Consider b_1, b_2, \dots to be the positive indeterminate coefficients. Let $N = 1, 2, \dots$ Put

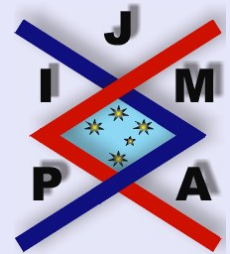
$$(2.9) \quad C_k = \sum_{n=k}^N \frac{n b_k^2}{(b_1 + b_2 + \cdots + b_n)^2}, \quad 1 \leq k \leq N.$$

Applying Lemma 2.2, we obtain

$$(2.10) \quad \sum_{n=1}^N \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sum_{n=1}^N \sum_{k=1}^n \frac{n a_k b_k^2}{(b_1 + b_2 + \cdots + b_n)^2} \\ = \sum_{k=1}^N \sum_{n=k}^N \frac{n a_k b_k^2}{(b_1 + b_2 + \cdots + b_n)^2} \\ = \sum_{k=1}^N C_k a_k.$$

Choosing $b_k = k, k = 1, 2, \dots$, we have from (2.9) that

$$(2.11) \quad C_k = \sum_{n=k}^N \frac{n k^2}{(1 + 2 + \cdots + n)^2} = 4k^2 \sum_{n=k}^N \frac{1}{n(n+1)^2}.$$



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 9 of 14

On the other hand, we have the equality

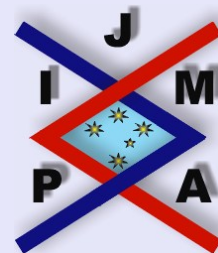
$$(2.12) \quad \frac{1}{2n^2 + \frac{2}{3}n} - \frac{1}{2(n+1)^2 + \frac{2}{3}(n+1)} - \frac{1}{n(n+1)^2} \\ = \frac{2}{n(n+1)^2(3n+1)(3n+4)} > 0, \quad \text{for all } n = 1, 2, \dots$$

Hence, it follows from (2.11) that

$$(2.13) \quad \sum_{n=k}^N \frac{1}{n(n+1)^2} \\ = \frac{1}{2k^2 + \frac{2}{3}k} - \frac{1}{2(N+1)^2 + \frac{2}{3}(N+1)} \\ - \sum_{n=k}^N \frac{2}{n(n+1)^2(3n+1)(3n+4)} \\ \leq \frac{1}{2k^2 + \frac{2}{3}k} - \sum_{n=k}^N \frac{2}{n(n+1)^2(3n+1)(3n+4)}, \quad 1 \leq k \leq N.$$

Hence, we obtain from (2.10), (2.13) that

$$(2.14) \quad C_k = 4k^2 \sum_{n=k}^N \frac{1}{n(n+1)^2} \\ \leq 2 - \frac{2}{3k+1} - 8k^2 \sum_{n=k}^N \frac{1}{n(n+1)^2(3n+1)(3n+4)}.$$



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

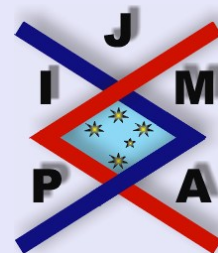
Quit

Page 10 of 14

$$\begin{aligned}
(2.15) \quad & \sum_{n=1}^N \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \\
& \leq \sum_{k=1}^N C_k a_k \\
& \leq \sum_{k=1}^N \left(2 - \frac{2}{3k+1} - 8k^2 \sum_{n=k}^N \frac{1}{n(n+1)^2(3n+1)(3n+4)} \right) a_k \\
& \leq \sum_{k=1}^N \left(2 - \frac{2}{3k+1} \right) a_k \\
& \quad - 8 \sum_{k=1}^N \left(\sum_{n=k}^N \frac{1}{n(n+1)^2(3n+1)(3n+4)} \right) k^2 a_k \\
& = \sum_{k=1}^N \left(2 - \frac{2}{3k+1} \right) a_k \\
& \quad - 8 \sum_{n=1}^N \left(\sum_{k=1}^n \frac{k^2 a_k}{n(n+1)^2(3n+1)(3n+4)} \right) \\
& = \sum_{k=1}^N \left(2 - \frac{2}{3k+1} \right) a_k - 8 \sum_{n=1}^N \beta_n.
\end{aligned}$$

where

$$(2.16) \quad \beta_n = \frac{\sum_{k=1}^n k^2 a_k}{n(n+1)^2(3n+1)(3n+4)}.$$



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

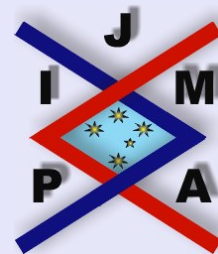
Page 11 of 14

We have

$$\begin{aligned}
 (2.17) \quad 0 < \beta_n &= \frac{\sum_{k=1}^n k^2 a_k}{n(n+1)^2(3n+1)(3n+4)} \\
 &\leq \frac{n^2 \sum_{k=1}^n a_k}{9n^5} \\
 &= \frac{1}{9n^3} \sum_{k=1}^n a_k \sim \frac{1}{9n^3} \sum_{k=1}^{\infty} a_k, \text{ as } n \rightarrow +\infty.
 \end{aligned}$$

Hence, the series $\sum_{n=1}^{\infty} \beta_n$ converges. Letting $N \rightarrow +\infty$ in (2.15), we obtain

$$\begin{aligned}
 (2.18) \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} &\leq \sum_{k=1}^{\infty} \left(2 - \frac{2}{3k+1}\right) a_k - 8 \sum_{n=1}^{\infty} \beta_n \\
 &= \sum_{k=1}^{\infty} \left(2 - \frac{2}{3k+1}\right) a_k \\
 &\quad - 8 \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{k^2 a_k}{n(n+1)^2(3n+1)(3n+4)} \right) \\
 &= \sum_{k=1}^{\infty} \left(2 - \frac{2}{3k+1}\right) a_k
 \end{aligned}$$



Note on the Carleman's Inequality for a Negative Power Number

Thanh Long Nguyen, Vu Duy Linh Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

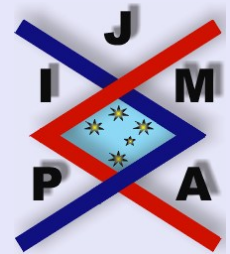
Close

Quit

Page 12 of 14

$$\begin{aligned}
& -8 \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \frac{1}{n(n+1)^2(3n+1)(3n+4)} \right) k^2 a_k \\
= & 2 \sum_{k=1}^{\infty} \left(1 - \frac{1}{3k+1} \right. \\
& \left. - 4k^2 \sum_{n=k}^{\infty} \frac{1}{n(n+1)^2(3n+1)(3n+4)} \right) a_k.
\end{aligned}$$

The proof of Theorem 2.1 is complete. □



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 13 of 14

References

- [1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, London, 1952.
- [2] BICHENG YANG AND L. DEBNATH, Some inequalities involving the constant e and an application to Carleman's inequality, *J. Math. Anal. Appl.*, **223** (1998), 347–353.
- [3] YAN PING AND GUOZHENG SUN, A strengthened Carleman's inequality, *J. Math. Anal. Appl.*, **240** (1999), 290–293.
- [4] XIAOJING YANG, On Carleman's inequality, *J. Math. Anal. Appl.*, **253** (2001), 691–694.
- [5] THANH LONG NGUYEN AND VU DUY LINH NGUYEN, The Carleman's inequality for a negative power number, *J. Math. Anal. Appl.*, **259** (2001), 219–225.



**Note on the Carleman's
Inequality for a Negative Power
Number**

Thanh Long Nguyen, Vu Duy Linh
Nguyen and Thi Thu Van Nguyen

Title Page

Contents



Go Back

Close

Quit

Page 14 of 14