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NOTE ON THE CARLEMAN'S INEQUALITY FOR A NEGATIVE POWER NUMBER

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Abstract

By the method of indeterminate coefficients we prove the inequality

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n,$$

where $a_n > 0, \ n = 1, 2, \dots \ \sum_{n=1}^{\infty} a_n < \infty$.

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Contents

1. Introduction

The following Carleman inequality is well known (see [1, Chapter 9.12]).

$$\sum_{n=1}^{\infty}\left(\frac{a_1^{\frac{1}{p}}+a_2^{\frac{1}{p}}+\cdots+a_n^{\frac{1}{p}}}{n}\right)^p \le \left(\frac{p}{p-1}\right)\sum_{n=1}^{\infty} a_n, \tag{1.1}$$

where $a_n \ge 0,\ n = 1, 2, \dots, \sum_{n=1}^{\infty} a_n < \infty$, and $p > 1$.

Letting $p \to +\infty$, it follows from (1.1) that

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} \le e\sum_{n=1}^{\infty} a_n. \tag{1.2}$$

In practice, the inequality (1.2) is strict; i.e.,

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty} a_n, \tag{1.3}$$

if $a_n \ge 0, n = 1, 2, \dots,\ 0 < \sum_{n=1}^{\infty} a_n < \infty$.

The constant e is sharp in the sense that it cannot be replaced by a smaller one.

Recently, the inequality (1.3) has also been improved by many authors, for example: Yang Bicheng and L. Debnath [2] with

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+2}\right) a_n, \tag{1.4}$$

in [3] by Yan Ping and Sun Guozheng with

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}\left(1+\frac{1}{n+1/5}\right)^{-1/2}a_n, \tag{1.5}$$

and in [4] by X. Yang with

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2(n+1)}-\frac{1}{24(n+1)^2}-\frac{1}{48(n+1)^3}\right)a_n, \tag{1.6}$$

and

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{6}\frac{b_k}{(n+1)^k}\right)a_n, \tag{1.7}$$

where

$$b_1=\frac{1}{2},\ b_2=\frac{1}{24},\ b_3=\frac{1}{48},\ b_4=\frac{73}{5760},\ b_5=\frac{11}{1280},\ b_6=\frac{1945}{580608},$$

$$a_n \geq 0,\ n=1,2,\dots,0<\sum_{n=1}^{\infty}a_n<\infty.$$

We rewrite the inequality (1.1) with $r=\frac{1}{p}$ as follows

$$\sum_{n=1}^{\infty}\left(\frac{a_1^r+a_2^r+\dots+a_n^r}{n}\right)^{1/r} \leq (1-r)^{-1/r}\sum_{n=1}^{\infty}a_n, \tag{1.8}$$

where $a_n \geq 0,\ n = 1, 2, \dots, \sum_{n=1}^{\infty} a_n < \infty$ and $0 < r < 1$.

In [5], we have improved Carleman's inequality (1.8) for a negative power number $r < 0$ as follows

$$\text{(1.9)} \quad \sum_{n=1}^{\infty} \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r} \leq \begin{cases} (1-r)^{-1/r} \sum_{n=1}^{\infty} a_n, & \text{if } -1 \leq r < 1, r \neq 0, \\ \frac{r}{r-1} 2^{(r-1)/r} \sum_{n=1}^{\infty} a_n, & \text{if } r < -1, \end{cases}$$

where $a_n > 0,\ n = 1, 2, \dots, \sum_{n=1}^{\infty} a_n < \infty$.

In the case of $r = -1$, we obtain from (1.9) the inequality

$$\text{(1.10)} \quad \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} a_n,$$

where $a_n > 0,\ n = 1, 2, \dots, \sum_{n=1}^{\infty} a_n < \infty$.

2. Main Result

In this paper, we shall prove the following theorem.

Theorem 2.1. *Let $a_n > 0,\ n = 1, 2, \ldots,$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then we have*

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - 4n^2 \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) a_n. \tag{2.1}$$

Remark 2.1. *From the inequality (2.1), we obtain the following inequalities:*

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right) a_n, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} \right) a_n, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} < 2 \sum_{n=1}^{\infty} a_n. \tag{2.4}$$

Indeed, we note that the inequalities (2.2), (2.3), (2.4) are implied from (2.1), because

$$1-\frac{1}{3n+1}-4n^2\sum_{k=n}^{\infty}\frac{1}{k(k+1)^2(3k+1)(3k+4)} \tag{2.5}$$
$$<1-\frac{1}{3n+1}-\frac{4n}{(n+1)^2(3n+1)(3n+4)}$$
$$<1-\frac{1}{3n+1}<1.$$

Hence, we obtain from (2.1), (2.5) that

$$\sum_{n=1}^{\infty}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \tag{2.6}$$
$$\le 2\sum_{n=1}^{\infty}\left(1-\frac{1}{3n+1}-4n^2\sum_{k=n}^{\infty}\frac{1}{k(k+1)^2(3k+1)(3k+4)}\right)a_n$$
$$=2\sum_{n=1}^{\infty}\left(1-\frac{1}{3n+1}-\frac{4n}{(n+1)^2(3n+1)(3n+4)}\right.$$
$$\left.-4n^2\sum_{k=n+1}^{\infty}\frac{1}{k(k+1)^2(3k+1)(3k+4)}\right)a_n$$
$$=2\sum_{n=1}^{\infty}\left(1-\frac{1}{3n+1}-\frac{4n}{(n+1)^2(3n+1)(3n+4)}\right)a_n$$

$$
\begin{aligned}
&- 8 \sum_{n=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2(3k+1)(3k+4)} \right) n^2 a_n \\
&< 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} - \frac{4n}{(n+1)^2(3n+1)(3n+4)} \right) a_n \\
&< 2 \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n+1} \right) a_n \\
&< 2 \sum_{n=1}^{\infty} a_n .
\end{aligned}
$$

To prove Theorem 2.1, we first prove the following lemma.

Lemma 2.2. *We have*

$$
\frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \frac{a_1 b_1^2 + a_2 b_2^2 + \cdots + a_n b_n^2}{(b_1 + b_2 + \cdots + b_n)^2}, \tag{2.7}
$$

where $a_k > 0,\ b_k > 0,\ \forall k = 1, 2, \ldots, n.$

Proof. This is a simple application of the Cauchy-Schwartz inequality

$$
\begin{aligned}
&(b_1 + b_2 + \cdots + b_n)^2 \\
&\quad \leq \left(\frac{1}{\sqrt{a_1}} \sqrt{a_1} b_1 + \frac{1}{\sqrt{a_2}} \sqrt{a_2} b_2 + \cdots + \frac{1}{\sqrt{a_n}} \sqrt{a_n} b_n \right)^2
\end{aligned} \tag{2.8}
$$

$$\leq \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(a_1 b_1^2 + a_2 b_2^2 + \cdots + a_n b_n^2\right).$$

□

Proof of Theorem 2.1. We prove the theorem by the method of indeterminate coefficients.

Consider $b_1, b_2, \ldots$ to be the positive indeterminate coefficients. Let $N = 1, 2, \ldots$ Put

$$C_k = \sum_{n=k}^{N} \frac{n b_k^2}{(b_1 + b_2 + \cdots + b_n)^2}, \quad 1 \leq k \leq N. \tag{2.9}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned} \sum_{n=1}^{N} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} &\leq \sum_{n=1}^{N} \sum_{k=1}^{n} \frac{n a_k b_k^2}{(b_1 + b_2 + \cdots + b_n)^2} \\ &= \sum_{k=1}^{N} \sum_{n=k}^{N} \frac{n a_k b_k^2}{(b_1 + b_2 + \cdots + b_n)^2} \\ &= \sum_{k=1}^{N} C_k a_k. \end{aligned} \tag{2.10}$$

Choosing $b_k = k$, $k = 1, 2, \ldots,$ we have from (2.9) that

$$C_k = \sum_{n=k}^{N} \frac{n k^2}{(1 + 2 + \cdots + n)^2} = 4k^2 \sum_{n=k}^{N} \frac{1}{n(n+1)^2}. \tag{2.11}$$

On the other hand, we have the equality

$$\frac{1}{2n^2+\frac{2}{3}n}-\frac{1}{2(n+1)^2+\frac{2}{3}(n+1)}-\frac{1}{n\left(n+1\right)^2} = \frac{2}{n\left(n+1\right)^2(3n+1)(3n+4)}>0, \quad \text{for all } n=1,2,\dots \tag{2.12}$$

Hence, it follows from (2.11) that

$$\begin{aligned}\sum_{n=k}^{N}\frac{1}{n\left(n+1\right)^2} &= \frac{1}{2k^2+\frac{2}{3}k}-\frac{1}{2(N+1)^2+\frac{2}{3}(N+1)} \\ &\quad -\sum_{n=k}^{N}\frac{2}{n\left(n+1\right)^2(3n+1)(3n+4)} \\ &\le \frac{1}{2k^2+\frac{2}{3}k}-\sum_{n=k}^{N}\frac{2}{n\left(n+1\right)^2(3n+1)(3n+4)}, \quad 1\le k\le N.\end{aligned} \tag{2.13}$$

Hence, we obtain from (2.10), (2.13) that

$$\begin{aligned}C_k &= 4k^2\sum_{n=k}^{N}\frac{1}{n\left(n+1\right)^2} \\ &\le 2-\frac{2}{3k+1}-8k^2\sum_{n=k}^{N}\frac{1}{n\left(n+1\right)^2(3n+1)(3n+4)}.\end{aligned} \tag{2.14}$$

$$(2.15)\quad \sum_{n=1}^{N} \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

$$\le \sum_{k=1}^{N} C_k a_k$$

$$\le \sum_{k=1}^{N}\left(2-\frac{2}{3k+1}-8k^2\sum_{n=k}^{N}\frac{1}{n\,(n+1)^2\,(3n+1)(3n+4)}\right)a_k$$

$$\le \sum_{k=1}^{N}\left(2-\frac{2}{3k+1}\right)a_k$$

$$-8\sum_{k=1}^{N}\left(\sum_{n=k}^{N}\frac{1}{n\,(n+1)^2\,(3n+1)(3n+4)}\right)k^2a_k$$

$$=\sum_{k=1}^{N}\left(2-\frac{2}{3k+1}\right)a_k$$

$$-8\sum_{n=1}^{N}\left(\sum_{k=1}^{n}\frac{k^2a_k}{n\,(n+1)^2\,(3n+1)(3n+4)}\right)$$

$$=\sum_{k=1}^{N}\left(2-\frac{2}{3k+1}\right)a_k-8\sum_{n=1}^{N}\beta_n.$$

where

$$(2.16)\quad \beta_n=\frac{\sum_{k=1}^{n}k^2a_k}{n\,(n+1)^2\,(3n+1)(3n+4)}.$$

We have

$$0 < \beta_n = \frac{\sum_{k=1}^{n} k^2 a_k}{n(n+1)^2(3n+1)(3n+4)} \leq \frac{n^2 \sum_{k=1}^{n} a_k}{9n^5} = \frac{1}{9n^3}\sum_{k=1}^{n} a_k \sim \frac{1}{9n^3}\sum_{k=1}^{\infty} a_k, \text{ as } n \to +\infty. \tag{2.17}$$

Hence, the series $\sum_{n=1}^{\infty} \beta_n$ converges. Letting $N \to +\infty$ in (2.15), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} &\leq \sum_{k=1}^{\infty}\left(2-\frac{2}{3k+1}\right)a_k - 8\sum_{n=1}^{\infty}\beta_n \\ &= \sum_{k=1}^{\infty}\left(2-\frac{2}{3k+1}\right)a_k \\ &\quad - 8\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{k^2 a_k}{n(n+1)^2(3n+1)(3n+4)}\right) \\ &= \sum_{k=1}^{\infty}\left(2-\frac{2}{3k+1}\right)a_k \end{aligned} \tag{2.18}$$

Note on the Carleman's Inequality for a Negative Power Number

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$$
\begin{aligned}
&\qquad - 8\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\frac{1}{n\left(n+1\right)^{2}\left(3n+1\right)\left(3n+4\right)}\right)k^{2}a_{k}\\
&= 2\sum_{k=1}^{\infty}\left(1-\frac{1}{3k+1}\right.\\
&\qquad\left.-4k^{2}\sum_{n=k}^{\infty}\frac{1}{n\left(n+1\right)^{2}\left(3n+1\right)\left(3n+4\right)}\right)a_{k}.
\end{aligned}
$$

The proof of Theorem 2.1 is complete. □

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