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INEQUALITIES IN $q-$FOURIER ANALYSIS

LAZHAR DHAOUADI, AHMED FITOUHI AND J. EL KAMEL

Ecole Préparatoire d'Ingénieur
Bizerte, Tunisia.
EMail: lazhardhaouadi@yahoo.fr

Faculté des Sciences de Tunis
1060 Tunis, Tunisia.
EMail: Ahmed.Fitouhi@fst.rnu.tn

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Abstract

In this paper we introduce the $q-$Bessel Fourier transform, the $q-$Bessel translation operator and the $q-$convolution product. We prove that the $q-$heat semigroup is contractive and we establish the $q-$analogue of Babenko inequalities associated to the $q-$Bessel Fourier transform. With applications and finally we enunciate a $q-$Bessel version of the central limit theorem.

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Contents

1. Introduction and Preliminaries

In introducing $q-$Bessel Fourier transforms, the $q-$Bessel translation operator and the $q-$convolution product we shall use the standard conventional notation as described in [4]. For further detailed information on $q-$derivatives, Jackson $q-$integrals and basic hypergeometric series we refer the interested reader to [4], [10], and [8].

The following two propositions will useful for the remainder of the paper.

Proposition 1.1. *Consider $0 < q < 1$. The series*

$$(w;q)_{\infty 1}\phi_1(0,w;q;z) = \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}} \frac{(wq^n;q)_\infty}{(q;q)_n} z^n,$$

defines an entire analytic function in z, w, which is also symmetric in z, w:

$$(w;q)_{\infty 1}\phi_1(0,w;q;z) = (z;q)_{\infty 1}\phi_1(0,z;q;w).$$

Both sides can be majorized by

$$|(w;q)_{\infty 1}\phi_1(0,w;q;z)| \leq (-|w|;q)_\infty(-|z|;q)_\infty.$$

Finally, for all $n \in \mathbb{N}$ we have

$$(q^{1-n};q)_{\infty 1}\phi_1(0,q^{1-n};q;z) = (-z)^n q^{\frac{n(n-1)}{2}} (q^{1+n};q)_{\infty 1}\phi_1(0,q^{1+n};q;q^n z).$$

Proof. See [10]. □

Now we introduce the following functional spaces:

$$\mathbb{R}_q = \{\mp q^n, \quad n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, \ n \in \mathbb{Z}\}.$$

Let $\mathcal{D}_q$, $\mathcal{C}_{q,0}$ and $\mathcal{C}_{q,b}$ denote the spaces of even smooth functions defined on $\mathbb{R}_q$ continuous at 0, which are respectively with compact support, vanishing at infinity and bounded. These spaces are equipped with the topology of uniform convergence, and by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_q x\right]^{\frac{1}{p}} < \infty.$$

We denote by $\mathcal{S}_q$ the $q-$analogue of the Schwartz space of even function f defined on $\mathbb{R}_q$ such that $D_q^k f$ is continuous at 0, and for all $n \in \mathbb{N}$ there is C_n such that

$$|D_q^k f(x)| \leq \frac{C_n}{(1+x^2)^n}, \quad \forall k \in \mathbb{N}, \forall x \in \mathbb{R}_q^+.$$

Ar the end of this section we introduce the $q-$Bessel operator as follows

$$\Delta_{q,v} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v} f(qx)\right].$$

Proposition 1.2. *Given two functions f and g in $\mathcal{L}_{q,2,v}$ such that*

$$\Delta_{q,v} f, \Delta_{q,v} g \in \mathcal{L}_{q,2,v}$$

then

$$\int_0^\infty \Delta_{q,v} f(x) g(x) x^{2v+1} d_q x = \int_0^\infty f(x) \Delta_{q,v} g(x) x^{2v+1} d_q x.$$

2. The Normalized Hahn-Exton $q-$Bessel Function

The normalized Hahn-Exton $q-$Bessel function of order v is defined as

$$j_v(x,q) = \frac{(q,q)_\infty}{(q^{v+1},q)_\infty} x^{-v} J_v^{(3)}(x,q) = {}_1\phi_1(0,q^{v+1},q,qx^2), \quad \Re(v) > -1,$$

where $J_v^{(3)}(\cdot,q)$ is the Hahn-Exton $q-$bessel function, (see [12]).

Proposition 2.1. *The function*

$$x \mapsto j_v(\lambda x, q^2),$$

is a solution of the following $q-$difference equation

$$\Delta_{q,v} f(x) = -\lambda^2 f(x)$$

Proof. See [9]. □

In the following we put

$$c_{q,v} = \frac{1}{1-q} \cdot \frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

Proposition 2.2. *Let $n, m \in \mathbb{Z}$ and $n \neq m$, then we have*

$$c_{q,v}^2 \int_0^\infty j_v(q^n x, q^2) j_v(q^m x, q^2) x^{2v+1} d_q x = \frac{q^{-2n(v+1)}}{1-q} \delta_{nm}.$$

Proof. See [10]. □

Proposition 2.3.

$$|j_v(q^n, q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \begin{cases} 1 & \textit{if} \quad n \geq 0, \\ q^{n^2+(2v+1)n} & \textit{if} \quad n < 0. \end{cases}$$

Proof. Use Proposition 1.1. □

3. $q-$Bessel Fourier Transform

The $q-$Bessel Fourier transform $\mathcal{F}_{q,v}$ is defined as follows

$$\mathcal{F}_{q,v}(f)(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t.$$

Proposition 3.1. *The $q-$Bessel Fourier transform*

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \to \mathcal{C}_{q,0},$$

satisfying

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{C}_{q,0}} \leq B_{q,v} \|f\|_{q,1,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Proof. Use Proposition 2.3. □

Theorem 3.2. *Given $f \in \mathcal{L}_{q,1,v}$ then we have*

$$\mathcal{F}_{q,v}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

If $f \in \mathcal{L}_{q,1,v}$ and $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}$ then

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proof. Let $t, y \in \mathbb{R}_q^+$, we put

$$\delta_{q,v}(t,y) = \begin{cases} \frac{1}{(1-q)t^{2v+2}} & \text{if} \quad t = y, \\ 0 & \text{if} \quad t \neq y. \end{cases}$$

It is not hard to see that

$$\int_0^{\infty} f(t)\delta_{q,v}(t,y)t^{2v+1}d_qt = f(y).$$

By Proposition 2.2, we can write

$$c_{q,v}^2 \int_0^{\infty} j_v(yx,q^2)j_v(tx,q^2)x^{2v+1}d_qx = \delta_{q,v}(t,y), \quad \forall t,y \in \mathbb{R}_q^+,$$

which leads to the result. □

Corollary 3.3. *The transformation*

$$\mathcal{F}_{q,v} : \mathcal{S}_q \to \mathcal{S}_q,$$

is an isomorphism, and

$$\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.$$

Proof. The result is deduced from properties of the space $\mathcal{S}_q$. □

4. $q-$Bessel Translation Operator

We introduce the $q-$Bessel translation operator as follows:

$$T^v_{q,x}f(y) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}(f)(t)j_v(xt,q^2)j_v(yt,q^2)t^{2v+1}d_qt,$$
$$\forall x,y\in\mathbb{R}_q^+, \forall f\in\mathcal{L}_{q,1,v}.$$

Proposition 4.1. *For any function $f\in\mathcal{L}_{q,1,v}$ we have*

$$T^v_{q,x}f(y) = T^v_{q,y}f(x),$$

and

$$T^v_{q,x}f(0) = f(x).$$

Proposition 4.2. *For all $x,y\in\mathbb{R}_q^+$, we have*

$$T^v_{q,x}j_v(\lambda y,q^2) = j_v(\lambda x,q^2)j_v(\lambda y,q^2).$$

Proof. Use Proposition 2.2. □

Proposition 4.3. *Let $f\in\mathcal{L}_{q,1,v}$ then*

$$T^v_{q,x}f(y) = \int_0^\infty f(z)D_v(x,y,z)z^{2v+1}d_qz,$$

where

$$D_v(x,y,z) = c^2_{q,v}\int_0^\infty j_v(xt,q^2)j_v(yt,q^2)j_v(zt,q^2)t^{2v+1}d_qt.$$

Proof. Indeed,

$$\begin{aligned}
&T^v_{q,x} f(y) \\
&= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\
&= c_{q,v} \int_0^\infty \left[c_{q,v} \int_0^\infty f(z) j_v(zt, q^2) z^{2v+1} d_q t \right] j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\
&= \int_0^\infty f(z) \left[c^2_{q,v} \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t \right] z^{2v+1} d_q z,
\end{aligned}$$

which leads to the result. □

Proposition 4.4.

$$\lim_{z \to \infty} D_v(x, y, z) = 0$$

and

$$(1-q) \sum_{s \in \mathbb{Z}} q^{(2v+2)s} D_v(x, y, q^s) = 1$$

Proof. To prove the first relation use Proposition 3.1. The second identity is deduced from Proposition 4.2: if $f = 1$ then $T^v_{q,x} f = 1$. □

Proposition 4.5. *Given $f \in \mathcal{S}_q$ then*

$$T^v_{q,x} f(y) = \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} y^{2n} \Delta^n_{q,v} f(x).$$

Proof. By the use of Proposition 2.1 and the fact that

$$\Delta_{q,v}^n f(x) = (-1)^n c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) t^{2n} j_v(xt, q^2) t^{2v+1} d_q t.$$

□

Proposition 4.6. *If* $v = -\frac{1}{2}$ *then*

$$D_v(q^m, q^r, q^k) = \frac{q^{2(r-m)(k-m)-m}}{(1-q)(q;q)_\infty} (q^{2(r-m)+1}; q)_\infty {}_1\phi_1(0, q^{2(r-m)+1}, q; q^{2(k-m)+1}).$$

Proof. Indeed

$$\Delta_{q,v}^n = \frac{q^{-n(n+1)}}{x^{2n}} \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ k+n \end{bmatrix}_q (-1)^{k+n} q^{\frac{(k+n)(k+n+1)}{2} - 2kn} \Lambda_q^k,$$

and use Proposition 4.5. □

5. $q-$Convolution Product

In harmonic analysis the positivity of the translation operator is crucial. It plays a central role in establishing some useful results, such as the property of the convolution product. Thus it is natural to investigate when this property holds for $T^v_{q,x}$. In the following we put

$$Q_v = \{q \in [0,1], \quad T^v_{q,x} \quad \text{is positive for all} \quad x \in \mathbb{R}^+_q\}.$$

Recall that $T^v_{q,x}$ is said to be positive if $T^v_{q,x} f \geq 0$ for $f \geq 0$.

Proposition 5.1. *If $v = -\frac{1}{2}$ then*

$$Q_v = [0, q_0],$$

where q_0 is the first zero of the following function:

$$q \mapsto {}_1\phi_1(0, q, q, q).$$

Proof. The operator $T^v_{q,x}$ is positive if and only if

$$D_v(x, y, q^s) \geq 0, \quad \forall x, y, q^s \in \mathbb{R}^+_q.$$

We replace $\frac{x}{y}$ by q^r, and we can choose $r \in \mathbb{N}$, because

$$T^v_{q,x} f(y) = T^v_{q,y} f(x),$$

thus we get

$$(q^{1+2s}, q)_\infty {}_1\phi_1(0, q^{1+2s}, q, q^{1+2r}) = \sum_{n=0}^{\infty} B_n(s, r), \quad \forall r, s \in \mathbb{N},$$

where

$$B_n(s,r) = \prod_{i=1}^{2n} \frac{q^{2r+i}}{1-q^i} \prod_{i=2n+2}^{\infty} (1-q^{2s+i}) \times \left[(1-q^{2s+2n+1}) - \frac{q^{2r+2n+1}}{1-q^{2n+1}} \right], \quad \forall n \in \mathbb{N}^*,$$

and

$$B_0(s,r) = \prod_{i=2}^{\infty} (1-q^{2s+i}) \left[(1-q^{2s+1}) - \frac{q^{2r+1}}{1-q} \right],$$

which leads to the result. □

In the rest of this work we choose $q \in Q_v$.

Proposition 5.2. *Given $f \in \mathcal{L}_{q,1,v}$ then*

$$\int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.$$

The $q-$convolution product of both functions $f, g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_q \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

Proposition 5.3. *Given two functions $f, g \in \mathcal{L}_{q,1,v}$ then*

$$f *_q g \in \mathcal{L}_{q,1,v},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \mathcal{F}_{q,v}(g).$$

Proof. We have

$$\|f *_q g\|_{q,1,v} \leq \|f\|_{q,1,v}\|g\|_{q,1,v}.$$

On the other hand

$$\begin{aligned}\mathcal{F}_{q,v}(f *_q g)(\lambda) &= \int_0^\infty \left[\int_0^\infty f(x) T^v_{q,y} j_v(\lambda x, q^2) x^{2v+1} d_q x\right] g(y) y^{2v+1} d_q y \\ &= \mathcal{F}_{q,v}(f)(\lambda)\ \mathcal{F}_{q,v}(g)(\lambda).\end{aligned}$$

□

6. $q-$Heat Semigroup

The $q-$heat semigroup is defined by:

$$P^v_{q,t}f(x) = G^v(\cdot,t,q^2) *_q f(x)$$
$$= c_{q,v}\int_0^\infty T^v_{q,x}G^v(y,t,q^2)f(y)y^{2v+1}d_qy, \quad \forall f \in \mathcal{L}_{q,1,v}.$$

$G^v(\cdot,t,q^2)$ is the $q-$Gauss kernel of $P^v_{q,t}$

$$G^v(x,t,q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e\left(-\frac{q^{-2v}}{t}x^2, q^2\right).$$

and $e(\cdot,q)$ the q-exponential function defined by

$$e(z,q) = \sum_{n=0}^{\infty}\frac{z^n}{(q,q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1.$$

Proposition 6.1. *The $q-$Gauss kernel $G^v(\cdot,t,q^2)$ satisfying*

$$\mathcal{F}_{q,v}\left\{G^v(\cdot,t,q^2)\right\}(x) = e(-tx^2,q^2),$$

and

$$\mathcal{F}_{q,v}\left\{e(-ty^2,q^2)\right\}(x) = G^v(x,t,q^2).$$

Proof. In [5], the Ramanujan identity was proved

$$\sum_{s\in\mathbb{Z}}\frac{z^s}{(bq^{2s},q^2)_\infty} = \frac{\left(bz, \frac{q^2}{bz}, q^2, q^2\right)_\infty}{\left(b, z, \frac{q^2}{b}, q^2\right)_\infty},$$

which implies

$$\int_0^\infty e(-ty^2, q^2) y^{2n} y^{2v+1} d_q y = (1-q) \sum_s \frac{(q^{2n+2v+2})^s}{(-tq^{2s}, q^2)_\infty}$$
$$= (1-q) \frac{\left(-tq^{2n+2v+2}, -\frac{q^{-2n-2v}}{t}, q^2, q^2\right)_\infty}{\left(-t, q^{2n+2v+2}, -\frac{q^2}{t}, q^2\right)_\infty}.$$

The following identity leads to the result

$$(a, q^2)_\infty = (a, q^2)_n (q^{2n} a, q^2)_\infty,$$

and

$$(aq^{-2n}, q^2)_\infty = (-1)^n q^{-n^2+n} \left(\frac{a}{q^2}\right)^n \left(\frac{q^2}{a}, q^2\right)_n (a, q^2)_\infty.$$

□

Proposition 6.2. *For any functions $f \in \mathcal{S}_q$, we have*

$$P^v_{q,t} f(x) = e(t\Delta_{q,v}, q^2) f(x).$$

Proof. Indeed, if

$$c_{q,v} \int_0^\infty G^v(y, t, q^2) y^{2n} y^{2v+1} d_q y = (q^{2v+2}, q^2)_n q^{-n(n+n)} t^n,$$

then

$$P^v_{q,t}f(x) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2,q^2)_n(q^{2v+2},q^2)_n} \times \left[c_{q,v}\int_0^\infty G^v(y,t,q^2)y^{2n}y^{2v+1}d_qy\right]\Delta^n_{q,v}f(x).$$

□

Theorem 6.3. *For $f \in \mathcal{L}_{q,p,v}$ and $1 \le p < \infty$, we have*

$$\|P^v_{q,t}f\|_{q,p,v} \le \|f\|_{q,p,v}.$$

Proof. If $p = 1$ then

$$\|P^v_{q,t}f\|_{q,1,v} \le \|G^v(\cdot,t,q^2)\|_{q,1,v}\, \|f\|_{q,1,v} = \|f\|_{q,1,v}.$$

Now let $p > 1$ and we consider the following function

$$g : y \mapsto T^v_{q,x}G^v(y,t;q^2).$$

In addition

$$\|P^v_{q,t}f\|^p_{q,p} \le c^p_{q,v}\int_0^\infty \left[\int_0^\infty |f(y)g(y)|\, y^{2v+1}d_qy\right]^p x^{2v+1}d_qx.$$

By the use of the Hölder inequality and the fact that $\|G^v(\cdot,t,q^2)\|_{q,1,v} = \frac{1}{c_{q,v}}$, the result follows immediately. □

7. $q-$Wiener Algebra

For $u \in \mathcal{L}_{q,1,v}$ and $\lambda \in \mathbb{R}_q^+$, we introduce the following function

$$u_\lambda : x \mapsto \frac{1}{\lambda^{2v+2}} u\left(\frac{x}{\lambda}\right).$$

Proposition 7.1. *Given $u \in \mathcal{L}_{q,1,v}$ such that*

$$\int_0^\infty u(x) x^{2v+1} d_q x = 1,$$

then we have

$$\lim_{\lambda \to 0} \int_0^\infty f(x) u_\lambda(x) x^{2v+1} d_q x = f(0), \quad \forall f \in \mathcal{C}_{q,b}.$$

Corollary 7.2. *The following function*

$$G_\lambda^v : x \mapsto c_{q,v} G^v(x, \lambda^2, q^2),$$

checks the conditions of the preceding proposition.

Proof. Use Proposition 6.1. □

Theorem 7.3. *Given $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$, $1 \le p < \infty$ and f_λ defined by*

$$f_\lambda(x) = c_q \int_0^\infty \mathcal{F}_{q,v}(f)(y) e(-\lambda^2 y^2, q^2) j_v(xy, q^2) y^{2v+1} d_q y.$$

then we have

$$\lim_{\lambda \to 0} \|f - f_\lambda\|_{q,p,v} = 0.$$

Proof. We have

$$f *_q G_\lambda^v(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) e(-\lambda^2 t^2, q^2) j_v(tx, q^2) t^{2v+1} d_q t.$$

In addition, for all $\varepsilon > 0$, there exists a function $h \in \mathcal{L}_{q,p,v}$ with compact support in $[q^k, q^{-k}]$ such that

$$\|f - h\|_{q,p,v} < \varepsilon,$$

however

$$\|G_\lambda^v *_q f - f\|_{q,p,v} \leq \|G_\lambda^v *_q (f-h)\|_{q,p,v} + \|G_\lambda^v *_q h - h\|_{q,p,v} + \|f - h\|_{q,p,v}.$$

By Theorem 6.3 we get

$$\|G_\lambda^v *_q (f-h)\|_{q,p,v} \leq \|f - h\|_{q,p,v}.$$

Now, we will prove that

$$\lim_{\lambda \to 0} \|G_\lambda^v *_q h - h\|_{q,p,v} = 0.$$

Indeed, by the use of Corollary 7.2 we get

$$\lim_{\lambda \to 0} \int_0^1 |G_\lambda^v *_q h(x) - h(x)|^p x^{2v+1} d_q x = 0.$$

On the other hand the following function is decreasing on the interval $[1, \infty[$:

$$u \mapsto u^{2v+2} G^v(u).$$

If $\lambda < 1$, then we deduce that

$$T^v_{q,q^i} G^v_\lambda(x) \le T^v_{q,q^i} G(x).$$

We can use the dominated convergence theorem to prove that

$$\lim_{\lambda \to 0} \int_1^{\infty} |G^v_\lambda *_q h(x) - h(x)|^p x^{2v+1} d_q x = 0.$$

□

Corollary 7.4. *Given $f \in \mathcal{L}_{q,1,v}$ then*

$$f(x) = c_{q,v} \int_0^{\infty} \mathcal{F}_{q,v}(f)(y) j_v(xy, q^2) y^{2v+1} d_q y, \quad \forall x \in \mathbb{R}^+_q.$$

Proof. The result is deduced by Theorem 7.3 and the following relation

$$(1-q)x^{2v+2} |f(x) - f_\lambda(x)| \le \|f - f_\lambda\|_{q,1,v} \quad \forall x \in \mathbb{R}^+_q.$$

□

Now we attempt to study the $q-$Wiener algebra denoted by

$$\mathcal{A}_{q,v} = \left\{ f \in \mathcal{L}_{q,1,v}, \quad \mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v} \right\}.$$

Proposition 7.5. *For $1 \le p \le \infty$, we have*

1. $\mathcal{A}_{q,v} \subset \mathcal{L}_{q,p,v}$ *and* $\overline{\mathcal{A}_{q,v}} = \mathcal{L}_{q,p,v}$.

2. $\mathcal{A}_{q,v} \subset \mathcal{C}_{q,0}$ *and* $\overline{\mathcal{A}_{q,v}} = \mathcal{C}_{q,0}$.

Proof. 1. Given $h \in \mathcal{L}_{q,p,v}$ with compact support, and we put $h_n = h *_q G^v_{q^n}$. The function $h_n \in \mathcal{A}_{q,v}$ and by Theorem 7.3 we get

$$\lim_{n\to\infty} \|h - h_n\|_{q,p,v} = 0.$$

2. If $f \in \mathcal{C}_{q,0}$, then there exist $h \in \mathcal{C}_{q,0}$ with compact support on $[q^k, q^{-k}]$, such that

$$\|f - h\|_{\mathcal{C}_{q,0}} < \varepsilon,$$

and by Corollary 7.4 we prove that

$$\lim_{n\to\infty} \left[\sup_{x\in\mathbb{R}_q^+} |h(x) - h_n(x)| \right] = 0.$$

□

Theorem 7.6. *For $f \in \mathcal{L}_{q,2,v} \cap \mathcal{L}_{q,1,v}$, we have*

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proof. We put

$$f_n = f *_q G^v_{q^n},$$

which implies

$$\mathcal{F}_{q,v}(f_n)(t) = e(-q^{2n}t^2, q^2)\mathcal{F}_{q,v}(f)(t),$$

by Corollary 7.4 we get

$$f_n(x) = c_q \int_0^\infty \mathcal{F}_{q,v}(f_n)(t) j_v(tx, q^2) t^{2v+1} d_q t.$$

On the other hand

$$\int_0^\infty f(x) f_n(x) x^{2v+1} d_q x = \int_0^\infty \mathcal{F}_{q,v}(f)(x) \mathcal{F}_{q,v}(f_n)(x) x^{2v+1} d_q x.$$

Theorem 7.3 implies

$$\lim_{n\to\infty} \int_0^\infty \mathcal{F}_{q,v}(f)(x)^2 e(-q^{2n}x^2, q^2) x^{2v+1} d_q x = \|f\|_{q,2,v}^2.$$

The sequence $e(-q^{2n}x^2, q^2)$ is increasing. By the use of the Fatou-Beppo-Levi theorem we deduce the result. □

Theorem 7.7.

1. *The $q-$cosine Fourier transform $\mathcal{F}_{q,v}$ possesses an extension*

$$U : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}.$$

2. *For $f \in \mathcal{L}_{q,2,v}$, we have*

$$\|U(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

3. *The application U is bijective and*

$$U^{-1} = U.$$

Proof. Let the maps

$$u : \mathcal{A}_{q,v} \to \mathcal{A}_{q,v}, \quad f \mapsto \mathcal{F}_{q,v}(f).$$

Theorem 3.2 implies

$$\|u(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

The map u is uniformly continuous, with values in complete space $\mathcal{L}_{q,2,v}$. It has a prolongation U on $\overline{\mathcal{A}_{q,v}} = \mathcal{L}_{q,2,v}$. □

Proposition 7.8. *Given* $1 < p \leq 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*, if* $f \in \mathcal{L}_{q,p,v}$*, then* $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,p',v}$*,*

$$\|\mathcal{F}_{q,v}(f)\|_{q,p',v} \leq B_{p,q,v} \|f\|_{q,p,v},$$

where

$$B_{p,q,v} = B_{q,v}^{(\frac{2}{p}-1)}.$$

Proof. The result is a consequence of Proposition 3.1, Theorem 7.7 and the Riesz-Thorin theorem, see [13]. □

As an immediate consequence of Proposition 7.8, we have the following theorem:

Theorem 7.9. *Given* $1 < p, p', r \leq 2$ *and*

$$\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r},$$

if $f \in \mathcal{L}_{q,p,v}$ *and* $g \in \mathcal{L}_{q,p',v}$*, then*

$$f *_q g \in \mathcal{L}_{q,r,v},$$

and

$$\|f *_q g\|_{q,r,v} \leq B_{q,p,v} B_{q,p',v} B_{q,r',v} \|f\|_{q,p,v} \|g\|_{q,p',v},$$

where

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Proof. We can write

$$f *_q g = \mathcal{F}_{q,v} \left\{ \mathcal{F}_{q,v}(f) \mathcal{F}_{q,v}(g) \right\},$$

the use of Proposition 7.8 and the Hölder inequality leads to the result. □

Now we are in a position to establish the hypercontractivity of the $q-$heat semigroup $P^v_{q,t}$. For more information about this notion, the reader can consult ([1, 2, 3]).

Proposition 7.10. *For $f \in \mathcal{L}_{q,p',v}$ and $t \in \mathbb{R}^+_q$, we have*

$$\|P^v_{q,t} f\|_{q,p,v} \leq B_{q,p',v} B_{q,p_1,v} c(r,q,v) t^{-\frac{v+1}{r}} \|f\|_{q,p',v},$$

where

$$1 < p' < p \leq 2, \quad \frac{1}{p} + \frac{1}{p_1} = 1, \quad \frac{1}{r} = \frac{1}{p'} - \frac{1}{p},$$

and

$$c(r,q,v) = \|e(-x^2, q^2)\|_{q,r,v}.$$

Proof. The result is deduced by the following relations

$$\mathcal{F}_{q,v} \left\{ G^v(\cdot, t, q^2) \right\}(x) = e(-tx^2, q^2),$$

and

$$\|\mathcal{F}_{q,v} \left\{ G^v(\cdot, t, q^2) \right\}\|_{q,r,v} = c(r,q,v) t^{-\frac{v+1}{r}}.$$

□

8. $q-$Central Limit Theorem

In this section we study the analogoue of the well known central limit theorem with the aid of the $q-$Bessel Fourier transform.

For this, we consider the set $\mathcal{M}_q^+$ of positive and bounded measures on $\mathbb{R}_q^+$. The q-cosine Fourier transform of $\xi \in \mathcal{M}_q^+$ is defined by

$$\mathcal{F}_{q,v}(\xi)(x) = \int_0^\infty j_v(tx, q^2) t^{2v+1} d_q\xi(t).$$

The $q-$convolution product of two measures $\xi, \rho \in \mathcal{M}_q^+$ is given by

$$\xi *_q \rho(f) = \int_0^\infty T_{q,x}^v f(t) t^{2v+1} d_q\xi(x) d_q\rho(t),$$

and we have

$$\mathcal{F}_{q,v}(\xi *_q \rho) = \mathcal{F}_{q,v}(\xi)\mathcal{F}_{q,v}(\rho).$$

We begin by showing the following result

Proposition 8.1. *For $f \in \mathcal{A}_{q,v}$ and $\xi \in \mathcal{M}_q^+$, we have*

$$\int_0^\infty f(x) x^{2v+1} d_q\xi(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x) x^{2v+1} d_q x.$$

As a direct consequence we may state

Corollary 8.2. *Given $\xi, \xi' \in \mathcal{M}_q^+$ such that*

$$\mathcal{F}_{q,v}(\xi) = \mathcal{F}_{q,v}(\xi'),$$

then $\xi = \xi'$.

Proof. By Proposition 8.1, we have

$$\int_0^\infty f(x)x^{2v+1}d_q\xi(x) = \int_0^\infty f(x)x^{2v+1}d_q\xi'(x), \quad \forall f \in \mathcal{A}_{q,v}.$$

from the assertion (2) of Proposition 7.5, we conclude that $\xi = \xi'$. □

Theorem 8.3. *Let $(\xi_n)_{n\geq 0}$ be a sequence of probability measures of $\mathcal{M}_q^+$ such that*

$$\lim_{n\to\infty} \mathcal{F}_{q,v}(\xi_n)(t) = \psi(t),$$

then there exists $\xi \in \mathcal{M}_q^+$ such that the sequence ξ_n converges strongly toward ξ, and

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

Proof. We consider the map I_n defined by

$$I_n(u) = \int_0^\infty u(x)x^{2v+1}d_q\xi_n(x), \quad \forall f \in \mathcal{C}_{q,0}.$$

By the following inequality

$$|I_n(u)| \leq \|u\|_{C_{q,0}},$$

and by Proposition 8.1, we get

$$I_n(f) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi_n)(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$

which implies

$$\lim_{n\to\infty} I_n(f) = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v}.$$

On the other hand, by assertion (2) of Proposition 7.5, and by the use of the Ascoli theorem (see [11]):

Consider a sequence of equicontinuous linear forms on $\mathcal{C}_{q,0}$ which converge on a dense part $\mathcal{A}_{q,v}$ then converge on the entire $\mathcal{C}_{q,0}$. We get

$$\lim_{n\to\infty} I_n(u) = \int_0^\infty \mathcal{F}_{q,v}(u)(x)\psi(x)x^{2v+1}d_qx, \quad \forall u \in \mathcal{C}_{q,0}.$$

Finally there exist $\xi \in \mathcal{M}_q^+$ such that

$$\lim_{n\to\infty} \int_0^\infty u(x)x^{2v+1}d_q\xi_n(x) = \int_0^\infty u(x)x^{2v+1}d_q\xi(x), \quad \forall u \in \mathcal{C}_{q,0}.$$

On the other hand

$$\mathcal{F}_{q,v}(\mathcal{A}_{q,v}) = \mathcal{A}_{q,v},$$

and

$$\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x)d_qx = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$

which implies

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

□

Proposition 8.4. *Given $\xi \in \mathcal{M}_q^+$, and supposing that*

$$\sigma = \int_0^\infty t^2 t^{2v+1} d_q\xi(t) < \infty,$$

then

$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1} x^2 + o(x^2).$$

Proof. We write

$$j_v(tx,q^2) = 1 - \frac{q^2t^2}{(q^2,q^2)_1(q^{2v+2},q^2)_1} x^2 + x^2\theta(tx)t^2,$$

where

$$\lim_{x\to 0}\theta(x) = 0,$$

then

$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1} x^2 + \left[\int_0^\infty t^2\theta(tx)t^{2v+1} d_q\xi(t)\right] x^2.$$

□

Now we are in a position to present the $q-$central limit theorem.

Theorem 8.5. *Let $(\xi_n)_{n\geq 0}$ be a sequence of probability measures of $\mathcal{M}_q^+$ of total mass 1, satisfying*

$$\lim_{n\to\infty} n\sigma_n = \sigma, \quad \textit{where} \quad \sigma_n = \int_0^\infty t^2 t^{2v+1} d_q\xi_n(t),$$

and

$$\lim_{n\to\infty} n\widetilde{\sigma}_n = 0, \quad \textit{where} \quad \widetilde{\sigma}_n = \int_0^\infty \frac{t^4}{1+t^2} t^{2v+1} d_q\xi_n(t),$$

then ξ_n^{*n} *converge strongly toward a measure* ξ *defined by*

$$d_q\xi(x) = c_{q,v}\mathcal{F}_{q,v}\left(e^{-\frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1}t^2}\right)(x)d_qx.$$

Proof. We have

$$\mathcal{F}_{q,v}(\xi_n^{*n}) = (\mathcal{F}_{q,v}(\xi_n))^n,$$

and

$$\mathcal{F}_{q,v}(\xi_n)(x) = 1 - \frac{q^2\sigma_n}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + \theta_n(x)x^2,$$

where

$$\theta_n(x) = \int_0^\infty t^2\theta(tx)t^{2v+1}d_q\xi_n(t).$$

Consequently

$$(\mathcal{F}_{q,v}(\xi_n))^n(x) = \exp\left[n\log\left[1 - \frac{q^2\sigma_n}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + \theta_n(x)x^2\right]\right].$$

By the following inequality

$$|t^2\theta(tx)| \le C_x\frac{t^4}{1+t^2}, \quad \forall t\in\mathbb{R}_q^+,$$

where C_x is some constant, the result follows immediately. □

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