

# Journal of Inequalities in Pure and Applied Mathematics

## INEQUALITIES IN $q$ -FOURIER ANALYSIS

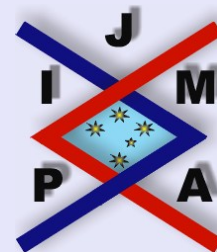
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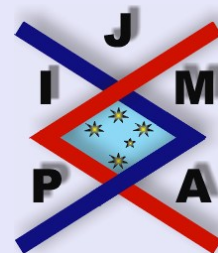
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## Abstract

In this paper we introduce the  $q$ -Bessel Fourier transform, the  $q$ -Bessel translation operator and the  $q$ -convolution product. We prove that the  $q$ -heat semigroup is contractive and we establish the  $q$ -analogue of Babenko inequalities associated to the  $q$ -Bessel Fourier transform. With applications and finally we enunciate a  $q$ -Bessel version of the central limit theorem.

*2000 Mathematics Subject Classification:* Primary 26D15, 26D20; Secondary 26D10.

*Key words:* Čebyšev functional, Grüss inequality, Bessel, Beta and Zeta function bounds.

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# 1. Introduction and Preliminaries

In introducing  $q$ -Bessel Fourier transforms, the  $q$ -Bessel translation operator and the  $q$ -convolution product we shall use the standard conventional notation as described in [4]. For further detailed information on  $q$ -derivatives, Jackson  $q$ -integrals and basic hypergeometric series we refer the interested reader to [4], [10], and [8].

The following two propositions will be useful for the remainder of the paper.

**Proposition 1.1.** Consider  $0 < q < 1$ . The series

$$(w; q)_{\infty} \phi_1(0, w; q; z) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} \frac{(wq^n; q)_{\infty}}{(q; q)_n} z^n,$$

defines an entire analytic function in  $z, w$ , which is also symmetric in  $z, w$ :

$$(w; q)_{\infty} \phi_1(0, w; q; z) = (z; q)_{\infty} \phi_1(0, z; q; w).$$

Both sides can be majorized by

$$|(w; q)_{\infty} \phi_1(0, w; q; z)| \leq (-|w|; q)_{\infty} (-|z|; q)_{\infty}.$$

Finally, for all  $n \in \mathbb{N}$  we have

$$(q^{1-n}; q)_{\infty} \phi_1(0, q^{1-n}; q; z) = (-z)^n q^{\frac{n(n-1)}{2}} (q^{1+n}; q)_{\infty} \phi_1(0, q^{1+n}; q; q^n z).$$

*Proof.* See [10]. □



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Now we introduce the following functional spaces:

$$\mathbb{R}_q = \{\mp q^n, n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}.$$

Let  $\mathcal{D}_q$ ,  $\mathcal{C}_{q,0}$  and  $\mathcal{C}_{q,b}$  denote the spaces of even smooth functions defined on  $\mathbb{R}_q$  continuous at 0, which are respectively with compact support, vanishing at infinity and bounded. These spaces are equipped with the topology of uniform convergence, and by  $\mathcal{L}_{q,p,v}$  the space of even functions  $f$  defined on  $\mathbb{R}_q$  such that

$$\|f\|_{q,p,v} = \left[ \int_0^\infty |f(x)|^p x^{2v+1} d_q x \right]^{\frac{1}{p}} < \infty.$$

We denote by  $\mathcal{S}_q$  the  $q$ -analogue of the Schwartz space of even function  $f$  defined on  $\mathbb{R}_q$  such that  $D_q^k f$  is continuous at 0, and for all  $n \in \mathbb{N}$  there is  $C_n$  such that

$$|D_q^k f(x)| \leq \frac{C_n}{(1+x^2)^n}, \quad \forall k \in \mathbb{N}, \forall x \in \mathbb{R}_q^+.$$

At the end of this section we introduce the  $q$ -Bessel operator as follows

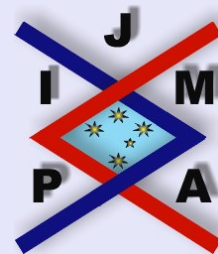
$$\Delta_{q,v} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v}f(qx)].$$

**Proposition 1.2.** *Given two functions  $f$  and  $g$  in  $\mathcal{L}_{q,2,v}$  such that*

$$\Delta_{q,v} f, \Delta_{q,v} g \in \mathcal{L}_{q,2,v}$$

then

$$\int_0^\infty \Delta_{q,v} f(x) g(x) x^{2v+1} d_q x = \int_0^\infty f(x) \Delta_{q,v} g(x) x^{2v+1} d_q x.$$



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## 2. The Normalized Hahn-Exton $q$ -Bessel Function

The normalized Hahn-Exton  $q$ -Bessel function of order  $v$  is defined as

$$j_v(x, q) = \frac{(q, q)_\infty}{(q^{v+1}, q)_\infty} x^{-v} J_v^{(3)}(x, q) = {}_1\phi_1(0, q^{v+1}, q, qx^2), \quad \Re(v) > -1,$$

where  $J_v^{(3)}(\cdot, q)$  is the Hahn-Exton  $q$ -bessel function, (see [12]).

**Proposition 2.1.** *The function*

$$x \mapsto j_v(\lambda x, q^2),$$

*is a solution of the following  $q$ -difference equation*

$$\Delta_{q,v} f(x) = -\lambda^2 f(x)$$

*Proof.* See [9]. □

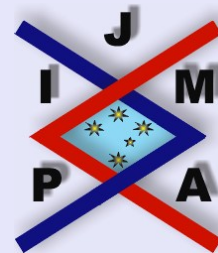
In the following we put

$$c_{q,v} = \frac{1}{1-q} \cdot \frac{(q^{2v+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

**Proposition 2.2.** *Let  $n, m \in \mathbb{Z}$  and  $n \neq m$ , then we have*

$$c_{q,v}^2 \int_0^\infty j_v(q^n x, q^2) j_v(q^m x, q^2) x^{2v+1} d_q x = \frac{q^{-2n(v+1)}}{1-q} \delta_{nm}.$$

*Proof.* See [10]. □



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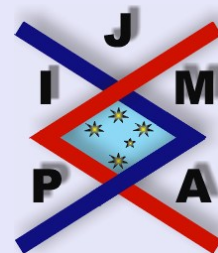
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### Proposition 2.3.

$$|j_v(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^{2v+2}; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0, \\ q^{n^2+(2v+1)n} & \text{if } n < 0. \end{cases}$$

*Proof.* Use Proposition 1.1. □



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### 3. $q$ -Bessel Fourier Transform

The  $q$ -Bessel Fourier transform  $\mathcal{F}_{q,v}$  is defined as follows

$$\mathcal{F}_{q,v}(f)(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t.$$

**Proposition 3.1.** *The  $q$ -Bessel Fourier transform*

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \rightarrow \mathcal{C}_{q,0},$$

satisfying

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{C}_{q,0}} \leq B_{q,v} \|f\|_{\mathcal{L}_{q,1,v}},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

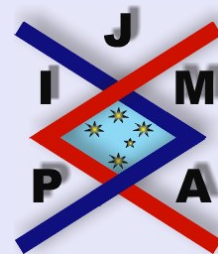
*Proof.* Use Proposition 2.3. □

**Theorem 3.2.** *Given  $f \in \mathcal{L}_{q,1,v}$  then we have*

$$\mathcal{F}_{q,v}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

*If  $f \in \mathcal{L}_{q,1,v}$  and  $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}$  then*

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{L}_{q,2,v}} = \|f\|_{\mathcal{L}_{q,2,v}}.$$



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*Proof.* Let  $t, y \in \mathbb{R}_q^+$ , we put

$$\delta_{q,v}(t, y) = \begin{cases} \frac{1}{(1-q)t^{2v+2}} & \text{if } t = y, \\ 0 & \text{if } t \neq y. \end{cases}$$

It is not hard to see that

$$\int_0^\infty f(t)\delta_{q,v}(t, y)t^{2v+1}d_qt = f(y).$$

By Proposition 2.2, we can write

$$c_{q,v}^2 \int_0^\infty j_v(yx, q^2)j_v(tx, q^2)x^{2v+1}d_qx = \delta_{q,v}(t, y), \quad \forall t, y \in \mathbb{R}_q^+,$$

which leads to the result. □

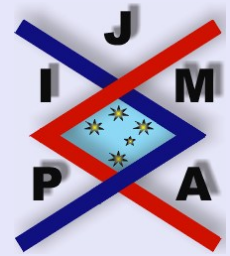
**Corollary 3.3.** *The transformation*

$$\mathcal{F}_{q,v} : \mathcal{S}_q \rightarrow \mathcal{S}_q,$$

*is an isomorphism, and*

$$\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.$$

*Proof.* The result is deduced from properties of the space  $\mathcal{S}_q$ . □



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## 4. $q$ -Bessel Translation Operator

We introduce the  $q$ -Bessel translation operator as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t,$$

$$\forall x, y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,1,v}.$$

**Proposition 4.1.** For any function  $f \in \mathcal{L}_{q,1,v}$  we have

$$T_{q,x}^v f(y) = T_{q,y}^v f(x),$$

and

$$T_{q,x}^v f(0) = f(x).$$

**Proposition 4.2.** For all  $x, y \in \mathbb{R}_q^+$ , we have

$$T_{q,x}^v j_v(\lambda y, q^2) = j_v(\lambda x, q^2) j_v(\lambda y, q^2).$$

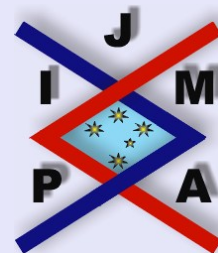
*Proof.* Use Proposition 2.2. □

**Proposition 4.3.** Let  $f \in \mathcal{L}_{q,1,v}$  then

$$T_{q,x}^v f(y) = \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z,$$

where

$$D_v(x, y, z) = c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t.$$



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*Proof.* Indeed,

$$\begin{aligned}
 T_{q,x}^v f(y) &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\
 &= c_{q,v} \int_0^\infty \left[ c_{q,v} \int_0^\infty f(z) j_v(zt, q^2) z^{2v+1} d_q t \right] j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\
 &= \int_0^\infty f(z) \left[ c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t \right] z^{2v+1} d_q z,
 \end{aligned}$$

which leads to the result.  $\square$

**Proposition 4.4.**

$$\lim_{z \rightarrow \infty} D_v(x, y, z) = 0$$

and

$$(1 - q) \sum_{s \in \mathbb{Z}} q^{(2v+2)s} D_v(x, y, q^s) = 1$$

*Proof.* To prove the first relation use Proposition 3.1. The second identity is deduced from Proposition 4.2: if  $f = 1$  then  $T_{q,x}^v f = 1$ .  $\square$

**Proposition 4.5.** Given  $f \in \mathcal{S}_q$  then

$$T_{q,x}^v f(y) = \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} y^{2n} \Delta_{q,v}^n f(x).$$



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*Proof.* By the use of Proposition 2.1 and the fact that

$$\Delta_{q,v}^n f(x) = (-1)^n c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) t^{2n} j_v(xt, q^2) t^{2v+1} d_q t.$$

□

**Proposition 4.6.** *If  $v = -\frac{1}{2}$  then*

$$D_v(q^m, q^r, q^k) = \frac{q^{2(r-m)(k-m)-m}}{(1-q)(q; q)_\infty} (q^{2(r-m)+1}; q)_{\infty 1} \phi_1(0, q^{2(r-m)+1}, q; q^{2(k-m)+1}).$$

*Proof.* Indeed

$$\Delta_{q,v}^n = \frac{q^{-n(n+1)}}{x^{2n}} \sum_{k=-n}^n \left[ \begin{matrix} 2n \\ k+n \end{matrix} \right]_q (-1)^{k+n} q^{\frac{(k+n)(k+n+1)}{2} - 2kn} \Lambda_q^k,$$

and use Proposition 4.5.

□



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## 5. $q$ -Convolution Product

In harmonic analysis the positivity of the translation operator is crucial. It plays a central role in establishing some useful results, such as the property of the convolution product. Thus it is natural to investigate when this property holds for  $T_{q,x}^v$ . In the following we put

$$Q_v = \{q \in [0, 1], \quad T_{q,x}^v \text{ is positive for all } x \in \mathbb{R}_q^+\}.$$

Recall that  $T_{q,x}^v$  is said to be positive if  $T_{q,x}^v f \geq 0$  for  $f \geq 0$ .

**Proposition 5.1.** *If  $v = -\frac{1}{2}$  then*

$$Q_v = [0, q_0],$$

where  $q_0$  is the first zero of the following function:

$$q \mapsto {}_1\phi_1(0, q, q, q).$$

*Proof.* The operator  $T_{q,x}^v$  is positive if and only if

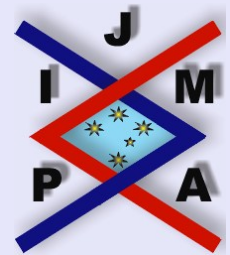
$$D_v(x, y, q^s) \geq 0, \quad \forall x, y, q^s \in \mathbb{R}_q^+.$$

We replace  $\frac{x}{y}$  by  $q^r$ , and we can choose  $r \in \mathbb{N}$ , because

$$T_{q,x}^v f(y) = T_{q,y}^v f(x),$$

thus we get

$$(q^{1+2s}, q)_{\infty} {}_1\phi_1(0, q^{1+2s}, q, q^{1+2r}) = \sum_{n=0}^{\infty} B_n(s, r), \quad \forall r, s \in \mathbb{N},$$



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where

$$B_n(s, r) = \prod_{i=1}^{2n} \frac{q^{2r+i}}{1-q^i} \prod_{i=2n+2}^{\infty} (1-q^{2s+i}) \\ \times \left[ (1-q^{2s+2n+1}) - \frac{q^{2r+2n+1}}{1-q^{2n+1}} \right], \quad \forall n \in \mathbb{N}^*,$$

and

$$B_0(s, r) = \prod_{i=2}^{\infty} (1-q^{2s+i}) \left[ (1-q^{2s+1}) - \frac{q^{2r+1}}{1-q} \right],$$

which leads to the result. □

In the rest of this work we choose  $q \in Q_v$ .

**Proposition 5.2.** *Given  $f \in \mathcal{L}_{q,1,v}$  then*

$$\int_0^{\infty} T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^{\infty} f(y) y^{2v+1} d_q y.$$

The  $q$ -convolution product of both functions  $f, g \in \mathcal{L}_{q,1,v}$  is defined by

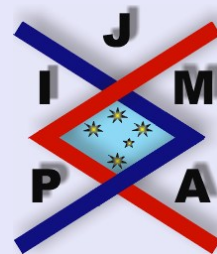
$$f *_q g(x) = c_q \int_0^{\infty} T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

**Proposition 5.3.** *Given two functions  $f, g \in \mathcal{L}_{q,1,v}$  then*

$$f *_q g \in \mathcal{L}_{q,1,v},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \mathcal{F}_{q,v}(g).$$



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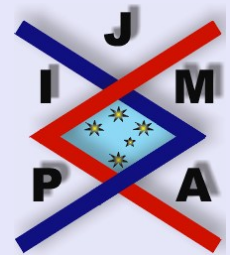
*Proof.* We have

$$\|f *_q g\|_{q,1,v} \leq \|f\|_{q,1,v} \|g\|_{q,1,v}.$$

On the other hand

$$\begin{aligned} \mathcal{F}_{q,v}(f *_q g)(\lambda) &= \int_0^\infty \left[ \int_0^\infty f(x) T_{q,y}^v j_v(\lambda x, q^2) x^{2v+1} d_q x \right] g(y) y^{2v+1} d_q y \\ &= \mathcal{F}_{q,v}(f)(\lambda) \mathcal{F}_{q,v}(g)(\lambda). \end{aligned}$$

□



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## 6. $q$ -Heat Semigroup

The  $q$ -heat semigroup is defined by:

$$\begin{aligned} P_{q,t}^v f(x) &= G^v(\cdot, t, q^2) *_q f(x) \\ &= c_{q,v} \int_0^\infty T_{q,x}^v G^v(y, t, q^2) f(y) y^{2v+1} d_q y, \quad \forall f \in \mathcal{L}_{q,1,v}. \end{aligned}$$

$G^v(\cdot, t, q^2)$  is the  $q$ -Gauss kernel of  $P_{q,t}^v$

$$G^v(x, t, q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e\left(-\frac{q^{-2v}}{t}x^2, q^2\right).$$

and  $e(\cdot, q)$  the  $q$ -exponential function defined by

$$e(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

**Proposition 6.1.** *The  $q$ -Gauss kernel  $G^v(\cdot, t, q^2)$  satisfying*

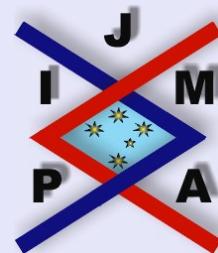
$$\mathcal{F}_{q,v} \{G^v(\cdot, t, q^2)\} (x) = e(-tx^2, q^2),$$

and

$$\mathcal{F}_{q,v} \{e(-ty^2, q^2)\} (x) = G^v(x, t, q^2).$$

*Proof.* In [5], the Ramanujan identity was proved

$$\sum_{s \in \mathbb{Z}} \frac{z^s}{(bq^{2s}, q^2)_\infty} = \frac{(bz, \frac{q^2}{bz}, q^2, q^2)_\infty}{(b, z, \frac{q^2}{b}, q^2)_\infty},$$



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which implies

$$\int_0^\infty e(-ty^2, q^2)y^{2n}y^{2v+1}d_qy = (1-q) \sum_s \frac{(q^{2n+2v+2})^s}{(-tq^{2s}, q^2)_\infty}$$

$$= (1-q) \frac{\left(-tq^{2n+2v+2}, -\frac{q^{-2n-2v}}{t}, q^2, q^2\right)_\infty}{\left(-t, q^{2n+2v+2}, -\frac{q^2}{t}, q^2\right)_\infty}.$$

The following identity leads to the result

$$(a, q^2)_\infty = (a, q^2)_n (q^{2n}a, q^2)_\infty,$$

and

$$(aq^{-2n}, q^2)_\infty = (-1)^n q^{-n^2+n} \left(\frac{a}{q^2}\right)^n \left(\frac{q^2}{a}, q^2\right)_n (a, q^2)_\infty.$$

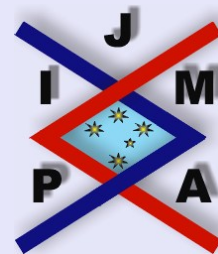
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**Proposition 6.2.** For any functions  $f \in \mathcal{S}_q$ , we have

$$P_{q,t}^v f(x) = e(t\Delta_{q,v}, q^2)f(x).$$

*Proof.* Indeed, if

$$c_{q,v} \int_0^\infty G^v(y, t, q^2)y^{2n}y^{2v+1}d_qy = (q^{2v+2}, q^2)_n q^{-n(n+n)}t^n,$$



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then

$$P_{q,t}^v f(x) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \times \left[ c_{q,v} \int_0^{\infty} G^v(y, t, q^2) y^{2n} y^{2v+1} d_q y \right] \Delta_{q,v}^n f(x).$$

□

**Theorem 6.3.** For  $f \in \mathcal{L}_{q,p,v}$  and  $1 \leq p < \infty$ , we have

$$\|P_{q,t}^v f\|_{q,p,v} \leq \|f\|_{q,p,v}.$$

*Proof.* If  $p = 1$  then

$$\|P_{q,t}^v f\|_{q,1,v} \leq \|G^v(\cdot, t, q^2)\|_{q,1,v} \|f\|_{q,1,v} = \|f\|_{q,1,v}.$$

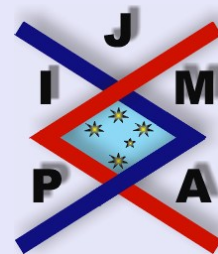
Now let  $p > 1$  and we consider the following function

$$g : y \mapsto T_{q,x}^v G^v(y, t; q^2).$$

In addition

$$\|P_{q,t}^v f\|_{q,p}^p \leq c_{q,v}^p \int_0^{\infty} \left[ \int_0^{\infty} |f(y)g(y)| y^{2v+1} d_q y \right]^p x^{2v+1} d_q x.$$

By the use of the Hölder inequality and the fact that  $\|G^v(\cdot, t, q^2)\|_{q,1,v} = \frac{1}{c_{q,v}}$ , the result follows immediately. □



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## 7. $q$ -Wiener Algebra

For  $u \in \mathcal{L}_{q,1,v}$  and  $\lambda \in \mathbb{R}_q^+$ , we introduce the following function

$$u_\lambda : x \mapsto \frac{1}{\lambda^{2v+2}} u\left(\frac{x}{\lambda}\right).$$

**Proposition 7.1.** *Given  $u \in \mathcal{L}_{q,1,v}$  such that*

$$\int_0^\infty u(x)x^{2v+1}d_qx = 1,$$

*then we have*

$$\lim_{\lambda \rightarrow 0} \int_0^\infty f(x)u_\lambda(x)x^{2v+1}d_qx = f(0), \quad \forall f \in \mathcal{C}_{q,b}.$$

**Corollary 7.2.** *The following function*

$$G_\lambda^v : x \mapsto c_{q,v}G^v(x, \lambda^2, q^2),$$

*checks the conditions of the preceding proposition.*

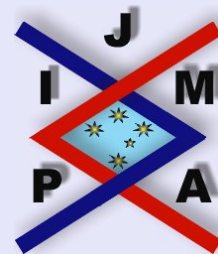
*Proof.* Use Proposition 6.1. □

**Theorem 7.3.** *Given  $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ ,  $1 \leq p < \infty$  and  $f_\lambda$  defined by*

$$f_\lambda(x) = c_q \int_0^\infty \mathcal{F}_{q,v}(f)(y)e(-\lambda^2 y^2, q^2)j_v(xy, q^2)y^{2v+1}d_qy.$$

*then we have*

$$\lim_{\lambda \rightarrow 0} \|f - f_\lambda\|_{q,p,v} = 0.$$



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*Proof.* We have

$$f *_q G_\lambda^v(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) e(-\lambda^2 t^2, q^2) j_v(tx, q^2) t^{2v+1} d_q t.$$

In addition, for all  $\varepsilon > 0$ , there exists a function  $h \in \mathcal{L}_{q,p,v}$  with compact support in  $[q^k, q^{-k}]$  such that

$$\|f - h\|_{q,p,v} < \varepsilon,$$

however

$$\|G_\lambda^v *_q f - f\|_{q,p,v} \leq \|G_\lambda^v *_q (f - h)\|_{q,p,v} + \|G_\lambda^v *_q h - h\|_{q,p,v} + \|f - h\|_{q,p,v}.$$

By Theorem 6.3 we get

$$\|G_\lambda^v *_q (f - h)\|_{q,p,v} \leq \|f - h\|_{q,p,v}.$$

Now, we will prove that

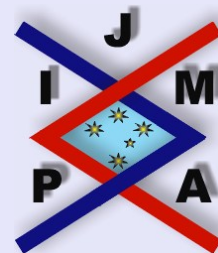
$$\lim_{\lambda \rightarrow 0} \|G_\lambda^v *_q h - h\|_{q,p,v} = 0.$$

Indeed, by the use of Corollary 7.2 we get

$$\lim_{\lambda \rightarrow 0} \int_0^1 |G_\lambda^v *_q h(x) - h(x)|^p x^{2v+1} d_q x = 0.$$

On the other hand the following function is decreasing on the interval  $[1, \infty[$ :

$$u \mapsto u^{2v+2} G^v(u).$$



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If  $\lambda < 1$ , then we deduce that

$$T_{q,q^i}^v G_\lambda^v(x) \leq T_{q,q^i}^v G(x).$$

We can use the dominated convergence theorem to prove that

$$\lim_{\lambda \rightarrow 0} \int_1^\infty |G_\lambda^v *_q h(x) - h(x)|^p x^{2v+1} d_q x = 0.$$

□

**Corollary 7.4.** Given  $f \in \mathcal{L}_{q,1,v}$  then

$$f(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(y) j_v(xy, q^2) y^{2v+1} d_q y, \quad \forall x \in \mathbb{R}_q^+.$$

*Proof.* The result is deduced by Theorem 7.3 and the following relation

$$(1 - q)x^{2v+2} |f(x) - f_\lambda(x)| \leq \|f - f_\lambda\|_{q,1,v} \quad \forall x \in \mathbb{R}_q^+.$$

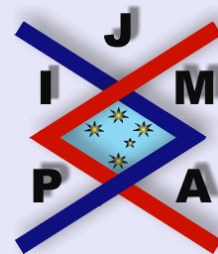
□

Now we attempt to study the  $q$ -Wiener algebra denoted by

$$\mathcal{A}_{q,v} = \{f \in \mathcal{L}_{q,1,v}, \mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}\}.$$

**Proposition 7.5.** For  $1 \leq p \leq \infty$ , we have

$$1. \mathcal{A}_{q,v} \subset \mathcal{L}_{q,p,v} \quad \text{and} \quad \overline{\mathcal{A}_{q,v}} = \mathcal{L}_{q,p,v}.$$



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$$2. \mathcal{A}_{q,v} \subset \mathcal{C}_{q,0} \quad \text{and} \quad \overline{\mathcal{A}_{q,v}} = \mathcal{C}_{q,0}.$$

*Proof.* 1. Given  $h \in \mathcal{L}_{q,p,v}$  with compact support, and we put  $h_n = h *_q G_{q^n}^v$ . The function  $h_n \in \mathcal{A}_{q,v}$  and by Theorem 7.3 we get

$$\lim_{n \rightarrow \infty} \|h - h_n\|_{q,p,v} = 0.$$

2. If  $f \in \mathcal{C}_{q,0}$ , then there exist  $h \in \mathcal{C}_{q,0}$  with compact support on  $[q^k, q^{-k}]$ , such that

$$\|f - h\|_{\mathcal{C}_{q,0}} < \varepsilon,$$

and by Corollary 7.4 we prove that

$$\lim_{n \rightarrow \infty} \left[ \sup_{x \in \mathbb{R}_q^+} |h(x) - h_n(x)| \right] = 0.$$

□

**Theorem 7.6.** For  $f \in \mathcal{L}_{q,2,v} \cap \mathcal{L}_{q,1,v}$ , we have

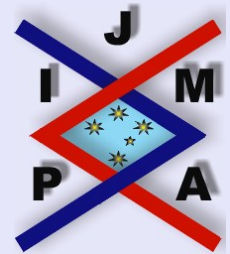
$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

*Proof.* We put

$$f_n = f *_q G_{q^n}^v,$$

which implies

$$\mathcal{F}_{q,v}(f_n)(t) = e(-q^{2n}t^2, q^2)\mathcal{F}_{q,v}(f)(t),$$



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by Corollary 7.4 we get

$$f_n(x) = c_q \int_0^\infty \mathcal{F}_{q,v}(f_n)(t) j_v(tx, q^2) t^{2v+1} d_q t.$$

On the other hand

$$\int_0^\infty f(x) f_n(x) x^{2v+1} d_q x = \int_0^\infty \mathcal{F}_{q,v}(f)(x) \mathcal{F}_{q,v}(f_n)(x) x^{2v+1} d_q x.$$

Theorem 7.3 implies

$$\lim_{n \rightarrow \infty} \int_0^\infty \mathcal{F}_{q,v}(f)(x)^2 e(-q^{2n} x^2, q^2) x^{2v+1} d_q x = \|f\|_{q,2,v}^2.$$

The sequence  $e(-q^{2n} x^2, q^2)$  is increasing. By the use of the Fatou-Beppo-Levi theorem we deduce the result.  $\square$

**Theorem 7.7.**

1. The  $q$ -cosine Fourier transform  $\mathcal{F}_{q,v}$  possesses an extension

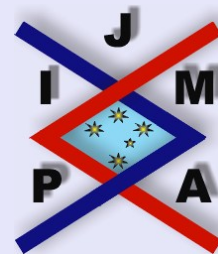
$$U : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}.$$

2. For  $f \in \mathcal{L}_{q,2,v}$ , we have

$$\|U(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

3. The application  $U$  is bijective and

$$U^{-1} = U.$$



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*Proof.* Let the maps

$$u : \mathcal{A}_{q,v} \rightarrow \mathcal{A}_{q,v}, \quad f \mapsto \mathcal{F}_{q,v}(f).$$

Theorem 3.2 implies

$$\|u(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

The map  $u$  is uniformly continuous, with values in complete space  $\mathcal{L}_{q,2,v}$ . It has a prolongation  $U$  on  $\overline{\mathcal{A}_{q,v}} = \mathcal{L}_{q,2,v}$ .  $\square$

**Proposition 7.8.** Given  $1 < p \leq 2$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , if  $f \in \mathcal{L}_{q,p,v}$ , then  $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,p',v}$ ,

$$\|\mathcal{F}_{q,v}(f)\|_{q,p',v} \leq B_{p,q,v} \|f\|_{q,p,v},$$

where

$$B_{p,q,v} = B_{q,v}^{\left(\frac{2}{p}-1\right)}.$$

*Proof.* The result is a consequence of Proposition 3.1, Theorem 7.7 and the Riesz-Thorin theorem, see [13].  $\square$

As an immediate consequence of Proposition 7.8, we have the following theorem:

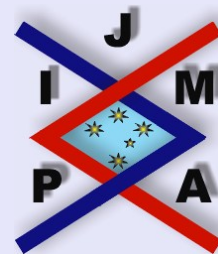
**Theorem 7.9.** Given  $1 < p, p', r \leq 2$  and

$$\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r},$$

if  $f \in \mathcal{L}_{q,p,v}$  and  $g \in \mathcal{L}_{q,p',v}$ , then

$$f *_q g \in \mathcal{L}_{q,r,v},$$

and



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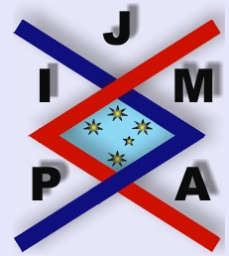


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$$\|f *_q g\|_{q,r,v} \leq B_{q,p,v} B_{q,p',v} B_{q,r',v} \|f\|_{q,p,v} \|g\|_{q,p',v},$$

where

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

*Proof.* We can write

$$f *_q g = \mathcal{F}_{q,v} \{ \mathcal{F}_{q,v}(f) \mathcal{F}_{q,v}(g) \},$$

the use of Proposition 7.8 and the Hölder inequality leads to the result.  $\square$

Now we are in a position to establish the hypercontractivity of the  $q$ -heat semigroup  $P_{q,t}^v$ . For more information about this notion, the reader can consult ([1, 2, 3]).

**Proposition 7.10.** For  $f \in \mathcal{L}_{q,p',v}$  and  $t \in \mathbb{R}_q^+$ , we have

$$\|P_{q,t}^v f\|_{q,p,v} \leq B_{q,p',v} B_{q,p_1,v} c(r, q, v) t^{-\frac{v+1}{r}} \|f\|_{q,p',v},$$

where

$$1 < p' < p \leq 2, \quad \frac{1}{p} + \frac{1}{p_1} = 1, \quad \frac{1}{r} = \frac{1}{p'} - \frac{1}{p},$$

and

$$c(r, q, v) = \|e(-x^2, q^2)\|_{q,r,v}.$$

*Proof.* The result is deduced by the following relations

$$\mathcal{F}_{q,v} \{ G^v(\cdot, t, q^2) \} (x) = e(-tx^2, q^2),$$

and

$$\|\mathcal{F}_{q,v} \{ G^v(\cdot, t, q^2) \} \|_{q,r,v} = c(r, q, v) t^{-\frac{v+1}{r}}.$$

$\square$



## 8. $q$ -Central Limit Theorem

In this section we study the analogue of the well known central limit theorem with the aid of the  $q$ -Bessel Fourier transform.

For this, we consider the set  $\mathcal{M}_q^+$  of positive and bounded measures on  $\mathbb{R}_q^+$ . The  $q$ -cosine Fourier transform of  $\xi \in \mathcal{M}_q^+$  is defined by

$$\mathcal{F}_{q,v}(\xi)(x) = \int_0^\infty j_v(tx, q^2)t^{2v+1}d_q\xi(t).$$

The  $q$ -convolution product of two measures  $\xi, \rho \in \mathcal{M}_q^+$  is given by

$$\xi *_q \rho(f) = \int_0^\infty T_{q,x}^v f(t)t^{2v+1}d_q\xi(x)d_q\rho(t),$$

and we have

$$\mathcal{F}_{q,v}(\xi *_q \rho) = \mathcal{F}_{q,v}(\xi)\mathcal{F}_{q,v}(\rho).$$

We begin by showing the following result

**Proposition 8.1.** For  $f \in \mathcal{A}_{q,v}$  and  $\xi \in \mathcal{M}_q^+$ , we have

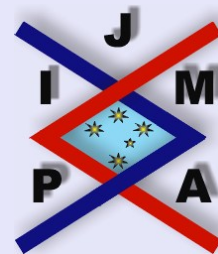
$$\int_0^\infty f(x)x^{2v+1}d_q\xi(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x)x^{2v+1}d_qx.$$

As a direct consequence we may state

**Corollary 8.2.** Given  $\xi, \xi' \in \mathcal{M}_q^+$  such that

$$\mathcal{F}_{q,v}(\xi) = \mathcal{F}_{q,v}(\xi'),$$

then  $\xi = \xi'$ .



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*Proof.* By Proposition 8.1, we have

$$\int_0^\infty f(x)x^{2v+1}d_q\xi(x) = \int_0^\infty f(x)x^{2v+1}d_q\xi'(x), \quad \forall f \in \mathcal{A}_{q,v}.$$

from the assertion (2) of Proposition 7.5, we conclude that  $\xi = \xi'$ . □

**Theorem 8.3.** *Let  $(\xi_n)_{n \geq 0}$  be a sequence of probability measures of  $\mathcal{M}_q^+$  such that*

$$\lim_{n \rightarrow \infty} \mathcal{F}_{q,v}(\xi_n)(t) = \psi(t),$$

*then there exists  $\xi \in \mathcal{M}_q^+$  such that the sequence  $\xi_n$  converges strongly toward  $\xi$ , and*

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

*Proof.* We consider the map  $I_n$  defined by

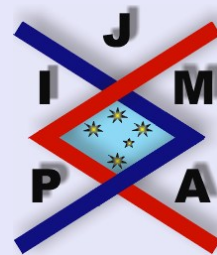
$$I_n(u) = \int_0^\infty u(x)x^{2v+1}d_q\xi_n(x), \quad \forall f \in \mathcal{C}_{q,0}.$$

By the following inequality

$$|I_n(u)| \leq \|u\|_{\mathcal{C}_{q,0}},$$

and by Proposition 8.1, we get

$$I_n(f) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi_n)(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$



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which implies

$$\lim_{n \rightarrow \infty} I_n(f) = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v}.$$

On the other hand, by assertion (2) of Proposition 7.5, and by the use of the Ascoli theorem (see [11]):

Consider a sequence of equicontinuous linear forms on  $\mathcal{C}_{q,0}$  which converge on a dense part  $\mathcal{A}_{q,v}$  then converge on the entire  $\mathcal{C}_{q,0}$ . We get

$$\lim_{n \rightarrow \infty} I_n(u) = \int_0^\infty \mathcal{F}_{q,v}(u)(x)\psi(x)x^{2v+1}d_qx, \quad \forall u \in \mathcal{C}_{q,0}.$$

Finally there exist  $\xi \in \mathcal{M}_q^+$  such that

$$\lim_{n \rightarrow \infty} \int_0^\infty u(x)x^{2v+1}d_q\xi_n(x) = \int_0^\infty u(x)x^{2v+1}d_q\xi(x), \quad \forall u \in \mathcal{C}_{q,0}.$$

On the other hand

$$\mathcal{F}_{q,v}(\mathcal{A}_{q,v}) = \mathcal{A}_{q,v},$$

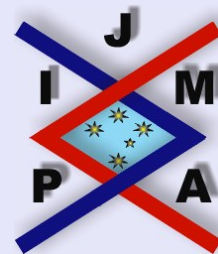
and

$$\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x)d_qx = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$

which implies

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

□



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**Proposition 8.4.** Given  $\xi \in \mathcal{M}_q^+$ , and supposing that

$$\sigma = \int_0^\infty t^2 t^{2v+1} d_q \xi(t) < \infty,$$

then

$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2 \sigma}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} x^2 + o(x^2).$$

*Proof.* We write

$$j_v(tx, q^2) = 1 - \frac{q^2 t^2}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} x^2 + x^2 \theta(tx) t^2,$$

where

$$\lim_{x \rightarrow 0} \theta(x) = 0,$$

then

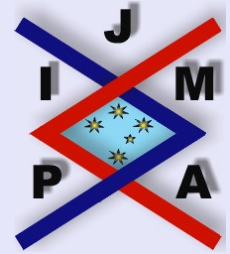
$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2 \sigma}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} x^2 + \left[ \int_0^\infty t^2 \theta(tx) t^{2v+1} d_q \xi(t) \right] x^2.$$

□

Now we are in a position to present the  $q$ -central limit theorem.

**Theorem 8.5.** Let  $(\xi_n)_{n \geq 0}$  be a sequence of probability measures of  $\mathcal{M}_q^+$  of total mass 1, satisfying

$$\lim_{n \rightarrow \infty} n \sigma_n = \sigma, \quad \text{where} \quad \sigma_n = \int_0^\infty t^2 t^{2v+1} d_q \xi_n(t),$$



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and

$$\lim_{n \rightarrow \infty} n\tilde{\sigma}_n = 0, \quad \text{where} \quad \tilde{\sigma}_n = \int_0^\infty \frac{t^4}{1+t^2} t^{2v+1} d_q \xi_n(t),$$

then  $\xi_n^{*n}$  converge strongly toward a measure  $\xi$  defined by

$$d_q \xi(x) = c_{q,v} \mathcal{F}_{q,v} \left( e^{-\frac{q^2 \sigma}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} t^2} \right) (x) d_q x.$$

*Proof.* We have

$$\mathcal{F}_{q,v}(\xi_n^{*n}) = (\mathcal{F}_{q,v}(\xi_n))^n,$$

and

$$\mathcal{F}_{q,v}(\xi_n)(x) = 1 - \frac{q^2 \sigma_n}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} x^2 + \theta_n(x) x^2,$$

where

$$\theta_n(x) = \int_0^\infty t^2 \theta(tx) t^{2v+1} d_q \xi_n(t).$$

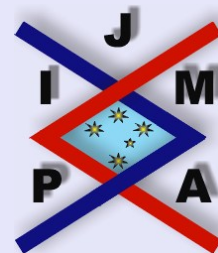
Consequently

$$(\mathcal{F}_{q,v}(\xi_n))^n(x) = \exp \left[ n \log \left[ 1 - \frac{q^2 \sigma_n}{(q^2, q^2)_1 (q^{2v+2}, q^2)_1} x^2 + \theta_n(x) x^2 \right] \right].$$

By the following inequality

$$|t^2 \theta(tx)| \leq C_x \frac{t^4}{1+t^2}, \quad \forall t \in \mathbb{R}_q^+,$$

where  $C_x$  is some constant, the result follows immediately. □



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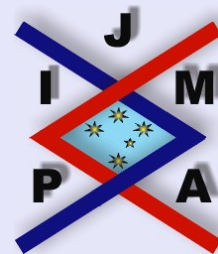
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