Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 4, Article 73, 2003

A VARIANT OF JENSEN'S INEQUALITY

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Received 14 August, 2003; accepted 27 August, 2003
Communicated by P.S. Bullen

ABSTRACT. If f is a convex function the following variant of the classical Jensen's Inequality is proved

$$f\left(x_1 + x_n - \sum w_k k_k\right) \leq f(x_1) + f(x_n) - \sum w_k f(x_k).$$

Key words and phrases: Jensen's inequality, Convex functions.

2000 *Mathematics Subject Classification.* 26D15.

1. MAIN THEOREM

Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and let w_k $(1 \leq k \leq n)$ be positive weights associated with these x_k and whose sum is unity. Then Jensen's inequality [2] reads :

Theorem 1.1. *If f is a convex function on an interval containing the x_k then*

$$f\left(\sum w_k x_k\right) \leq \sum w_k f(x_k). \tag{1.1}$$

Note: Here and, in all that follows, $\sum$ means $\sum_1^n$.

Our purpose in this note is to prove the following variant of (1.1).

Theorem 1.2. *If f is a convex function on an interval containing the x_k then*

$$f\left(x_1 + x_n - \sum w_k x_k\right) \leq f(x_1) + f(x_n) - \sum w_k f(x_k).$$

Towards proving this theorem we shall need the following lemma:

Lemma 1.3. *For f convex we have:*

$$f(x_1 + x_n - x_k) \leq f(x_1) + f(x_n) - f(x_k), \quad (1 \leq k \leq n). \tag{1.2}$$

ISSN (electronic): 1443-5756

116-03

2. THE PROOFS

Proof of Lemma 1.3. Write $y_k = x_1 + x_n - x_k$. Then $x_1 + x_n = x_k + y_k$ so that the pairs x_1, x_n and x_k, y_k possess the same mid-point. Since that is the case there exists λ such that

$$x_k = \lambda x_1 + (1-\lambda)x_n,$$
$$y_k = (1-\lambda)x_1 + \lambda x_n,$$

where $0 \le \lambda \le 1$ and $1 \le k \le n$.

Hence, applying (1.1) twice we get

$$\begin{aligned} f(y_k) &\le (1-\lambda)f(x_1) + \lambda f(x_n) \\ &= f(x_1) + f(x_n) - [\lambda f(x_1) + (1-\lambda)f(x_n)] \\ &\le f(x_1) + f(x_n) - f(\lambda x_1 + (1-\lambda)x_n) \\ &= f(x_1) + f(x_n) - f(x_k) \end{aligned}$$

and since $y_k = x_1 + x_n - x_k$ this concludes the proof of the lemma. □

Proof of Theorem 1.2. We have

$$\begin{aligned} f\left(x_1 + x_n - \sum w_k x_k\right) &= f\left(\sum w_k(x_1 + x_n - x_k)\right) \\ &\le \sum w_k f(x_1 + x_n - x_k) && \text{by (1.1)} \\ &\le \sum w_k [f(x_1) + f(x_n) - f(x_k)] && \text{by (1.2)} \\ &= f(x_1) + f(x_n) - \sum w f(x_k) \end{aligned}$$

and this concludes the proof. □

3. TWO EXAMPLES

Let us write $\widetilde{A} = x_1 + x_n - A$ and $\widetilde{G} = \frac{x_1 x_n}{G}$, where A and G denote the usual arithmetic and geometric means of the x_k.

(a) Then taking $f(x)$ as the convex function $-\log x$, Theorem 1.2 gives:

$$\widetilde{A} \ge \widetilde{G}$$

(b) Taking $f(x)$ as the function $\log \frac{1-x}{x}$ which is convex if $0 < x \le \frac{1}{2}$, Theorem 1.2 gives

$$\frac{\widetilde{A}(x)}{\widetilde{A}(1-x)} \ge \frac{\widetilde{G}(x)}{\widetilde{G}(1-x)}$$

provided that $x_k \in (0, \frac{1}{2}]$ for all k.

The example (a) is a special case of a family of inequalities found by a different method in [1]. The example (b) is, of course, an analogue of Ky-Fan's Inequality [2].

REFERENCES

[1] A.McD. MERCER, A monotonicity property of power means, *J. Ineq. Pure and Appl. Math.*, **3**(3) (2002), Article 40. [ONLINE: http://jipam.vu.edu.au/v3n3/014_02.html].

[2] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.