# HADAMARD PRODUCT OF CERTAIN MEROMORPHIC $p-$VALENT STARLIKE AND $p-$VALENT CONVEX FUNCTIONS

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| *Abstract:* | In this paper, we establish some results concerning the Hadamard product of certain meromorphic $p$-valent starlike and meromorphic $p$-valent convex functions analogous to those obtained by Vinod Kumar (J. Math. Anal. Appl. 113(1986), 230-234) and M. L. Mogra (Tamkang J. Math. 25(1994), no. 2, 157-162). |

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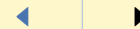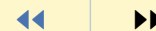
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# 1. Introduction

Throughout this paper, let $p \in \mathbb{N} = \{1, 2, \dots\}$ and let the functions of the form:

$$\varphi(z) = c_p z^p - \sum_{n=1}^{\infty} c_{p+n} z^{p+n} \qquad (c_p > 0; c_{p+n} \geq 0),$$

$$\Psi(z) = d_p z^p - \sum_{n=1}^{\infty} d_{p+n} z^{p+n} \qquad (d_p > 0; d_{p+n} \geq 0)$$

be regular and $p-$valent in the unit disc $U = \{z : |z| < 1\}$. Also, let

$$(1.1) \qquad f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (a_{p-1} > 0; a_{p+n-1} \geq 0),$$

$$f_i(z) = \frac{a_{p-1,i}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,i} z^{p+n-1} \qquad (a_{p-1,i} > 0; a_{p+n-1,i} \geq 0),$$
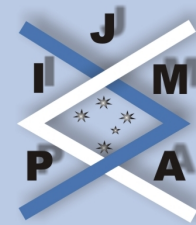
$$g(z) = \frac{b_{p-1}}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1} z^{p+n-1} \qquad (b_{p-1} > 0; b_{p+n-1} \geq 0)$$

and

$$g_j(z) = \frac{b_{p-1,j}}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1,j} z^{p+n-1} \qquad (b_{p-1,j} > 0; b_{p+n-1,j} \geq 0).$$

be regular and $p-$valent in the punctured disc $D = \{z : 0 < |z| < 1\}$.

Let $S_0^*(p, \alpha, \beta)$ denote the class of functions $\varphi(z)$ which satisfy the condition

$$\left| \frac{\frac{z\varphi'(z)}{\varphi(z)} - p}{\frac{z\varphi'(z)}{\varphi(z)} + p - 2\alpha} \right| < \beta$$

for some $\alpha, \beta$ $(0 \le \alpha < p,\ 0 < \beta \le 1,\ p \in \mathbb{N})$ and for all $z \in U$; and let $C_0(p, \alpha, \beta)$ be the class of functions $\varphi(z)$ for which $\frac{zf'(z)}{p} \in S_0^*(p, \alpha, \beta)$. It is well known that the functions in $S_0^*(p, \alpha, \beta)$ and $C_0^*(p, \alpha, \beta)$ are, respectively, $p-$valent starlike and $p-$valent convex functions of order $\alpha$ and type $\beta$ with negative coefficients in $U$ (see Aouf [1]).

Denote by $\Sigma S_0^*(p, \alpha, \beta)$, the class of functions $f(z)$ which satisfy the condition

$$(1.2) \qquad \left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} + 2\alpha - p} \right| < \beta$$

for some $\alpha, \beta$ $(0 \le \alpha < p,\ 0 < \beta \le 1,\ p \in \mathbb{N})$ and for all $z \in D$, and $\Sigma C_0^*(p, \alpha, \beta)$ be the class of functions $f(z)$ for which $\frac{-zf'(z)}{p} \in \Sigma S_0^*(p, \alpha, \beta)$. The functions in $\Sigma S_0^*(p, \alpha, \beta)$ and $\Sigma C_0^*(p, \alpha, \beta)$ are, respectively, called meromorphic $p-$valent starlike and meromorphic $p-$valent convex functions of order $\alpha$ and type $\beta$ with positive coefficients in $D$. The class $\Sigma S_0^*(p, \alpha, \beta)$ with $a_{p-1} = 1$ has been studied by Aouf [2] and Mogra [9].

Using similar arguments as given in ([2] and [9]), we can prove the following result for functions in $\Sigma S_0^*(p, \alpha, \beta)$.

A function $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$ if and only if

$$(1.3) \qquad \sum_{n=1}^{\infty} \left\{ [(n + 2p - 1) + \beta(n + 2\alpha - 1)]\, a_{n+p-1} \right\} \le 2\beta(p - \alpha) a_{p-1}.$$

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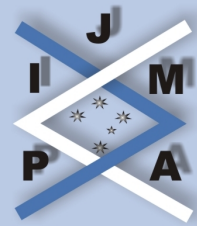
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The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2\beta(p-\alpha)a_{p-1}}{(n+2p-1)+\beta(n+2\alpha-1)}z^{p+n-1} \quad (p, n \in \mathbb{N}).$$

*Proof Outline.* Let $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$ be given by (1.1). Then, from (1.2) and (1.1), we have

$$(1.4) \quad \left| \frac{\sum_{n=1}^{\infty}(n+2p-1)a_{p+n-1}z^{2p+n-1}}{2(p-\alpha)a_{p-1} - \sum_{n=1}^{\infty}(n+2\alpha-1)a_{p+n-1}z^{2p+n-1}} \right| < \beta \quad (z \in U).$$

Since $|\mathrm{Re}(z)| \leq |z|$ $(z \in \mathbb{C})$, choosing $z$ to be real and letting $z \to 1^-$ through real values, (1.4) yields
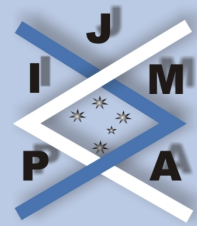
$$\sum_{n=1}^{\infty}(n+2p-1)a_{p+n-1} \leq 2\beta(p-\alpha)a_{p-1} - \sum_{n=1}^{\infty}\beta(n+2\alpha-1)a_{p+n-1},$$

which leads us to (1.3).

In order to prove the converse, we assume that the inequality (1.3) holds true. Then, if we let $z \in \partial U$, we find from (1.1) and (1.3) that

$$\left| \frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}+2\alpha-p} \right| \leq \frac{\sum_{n=1}^{\infty}(n+2p-1)a_{p+n-1}}{(p-\alpha)a_{p-1}-\sum_{n=1}^{\infty}(n+2\alpha-1)a_{p+n-1}}$$

$$< \beta \quad (z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}).$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Sigma S_0^*(p, \alpha, \beta)$. This completes the proof of (1.3). $\qquad \square$

Also we can prove that $f(z) \in \Sigma C_0^*(p, \alpha, \beta)$ if and only if

$$(1.5) \quad \sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right) \left[ (n+2p-1) + \beta(n+2\alpha-1) \right] a_{n+p-1} \right\}$$

$$\leq 2\beta(p-\alpha)a_{p-1} .$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2\beta(p-\alpha)a_{p-1}}{\left( \frac{n+2p-1}{p} \right) \left[ (n+2p-1) + \beta(n+2\alpha-1) \right]} z^{p+n-1} \quad (p, n \in \mathbb{N}) .$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa ([11], [12] and [13]), Kumar ([6], [7] and [8]), Aouf et al. [3], Hossen [5], Darwish [4] and Sekine [14]. Accordingly, the quasi-Hadamard product of two functions $\varphi(z)$ and $\Psi(z)$ is defined by
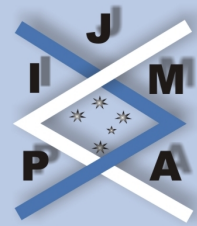
$$\varphi * \Psi(z) = c_p d_p z^p - \sum_{n=1}^{\infty} c_{p+n} d_{p+n} z^{p+n} .$$

Let us define the Hadamard product of two functions $f(z)$ and $g(z)$ by

$$f * g(z) = \frac{a_{p-1} b_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} b_{p+n-1} z^{p+n-1} .$$

Similarly, we can define the Hadamard product of more than two meromorphic $p-$valent functions.

We now introduce the following class of meromorphic $p-$valent functions in $D$.

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A function $f(z) \in \Sigma_k^*(p, \alpha, \beta)$ if and only if

$$(1.6) \quad \sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right)^k [(n+2p-1) + \beta(n+2\alpha-1)] a_{n+p-1} \right\}$$
$$\leq 2\beta(p-\alpha)a_{p-1}.$$

where $0 \leq \alpha < p$, $0 < \beta \leq 1$, $p \in \mathbb{N}$, and $k$ is any fixed nonnegative real number.

Evidently, $\Sigma_0^*(p, \alpha, \beta) \equiv \Sigma S_0^*(p, \alpha, \beta)$ and $\Sigma_1^*(p, \alpha, \beta) \equiv \Sigma C_0^*(p, \alpha, \beta)$. Further, $\Sigma_k^*(p, \alpha, \beta) \subset \Sigma_h^*(p, \alpha, \beta)$ if $k > h \geq 0$, the containment being proper. Moreover, for any positive integer $k$, we have the following inclusion relation

$$\Sigma_k^*(p, \alpha, \beta) \subset \Sigma_{k-1}^*(p, \alpha, \beta) \subset \cdots \subset \Sigma_2^*(p, \alpha, \beta) \subset \Sigma C_0^*(p, \alpha, \beta) \subset \Sigma S_0^*(p, \alpha, \beta).$$

We also note that for every nonnegative real number $k$, the class $\Sigma_k^*(p, \alpha, \beta)$ is nonempty as the functions

$$f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} \left( \frac{p+n-1}{p} \right)^{-k}$$
$$\times \left\{ \frac{2\beta(p-\alpha)}{(n+2p-1) + \beta(n+2\alpha-1)} \right\} a_{p-1} \lambda_{p+n-1} z^{p+n-1},$$

where $a_{p-1} > 0$, $0 \leq \alpha < p$, $0 < \beta \leq 1$, $p \in \mathbb{N}$, $a_{p-1} > 0$, $\lambda_{p+n-1} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n-1} \leq 1$, satisfy the inequality (1.1).

In this paper we establish certain results concerning the Hadamard product of meromorphic $p-$valent starlike and meromorphic $p-$valent convex functions of order $\alpha$ and type $\beta$ analogous to Kumar [7] and Mogra [10].
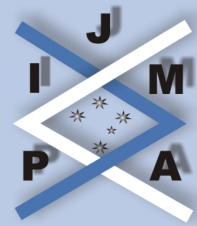
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## 2. The Main Theorems

**Theorem 2.1.** *Let the functions $f_i(z)$ belong to the class $\Sigma C_0^*(p, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\Sigma_{2m-1}^*(p, \alpha, \beta)$.*

*Proof.* It is sufficient to show that

$$\sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right)^{2m-1} [(n+2p-1) + \beta(n+2\alpha-1)] \prod_{i=1}^{m} a_{p+n-1,i} \right\}$$
$$\leq 2\beta(p-\alpha) \left[ \prod_{i=1}^{m} a_{p-1,i} \right] .$$

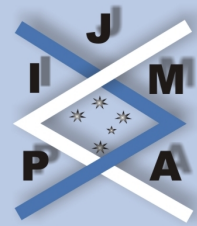Since $f_i(z) \in \Sigma C_0^*(p, \alpha, \beta)$, we have

$$(2.1) \quad \sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right) [(n+2p-1) + \beta(n+2\alpha-1)] a_{p+n-1,i} \right\}$$
$$\leq 2\beta(p-\alpha)a_{p-1,i},$$

for $i = 1, 2, \ldots, m$. Therefore,

$$\left( \frac{p+n-1}{p} \right) [(n+2p-1) + \beta(n+2\alpha-1)] a_{p+n-1,i} \leq 2\beta(p-\alpha)a_{p-1,i}$$

or

$$a_{p+n-1,i} \leq \left[ \frac{2\beta(p-\alpha)}{\left( \frac{p+n-1}{p} \right) [(n+2p-1) + \beta(n+2\alpha-1)]} \right] a_{p-1,i} ,$$

for every $i = 1, 2, \ldots, m$. The right-hand expression of the last inequality is not greater than $\left(\frac{p+n-1}{p}\right)^{-2} a_{p-1,i}$. Hence

$$(2.2) \qquad a_{p+n-1,i} \leq \left(\frac{p+n-1}{p}\right)^{-2} a_{p-1,i},$$
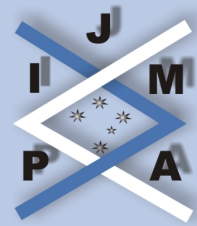
for every $i = 1, 2, \ldots, m$.

Using (2.2) for $i = 1, 2, \ldots, m-1$, and (2.1) for $i = m$, we obtain

$$\sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right)^{2m-1} [(n+2p-1) + \beta(n+2\alpha-1)] \prod_{i=1}^{m} a_{p+n-1,i} \right\}$$

$$\leq \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right)^{2m-1} [(n+2p-1) + \beta(n+2\alpha-1)] \right.$$

$$\left. \times \left( \left(\frac{p+n-1}{p}\right)^{-2(m-1)} \cdot \prod_{i=1}^{m-1} a_{p-1,i} \right) a_{p+n-1,m} \right\}$$

$$= \left[ \prod_{i=1}^{m-1} a_{p-1,i} \right] \sum_{n=1}^{\infty} \left\{ \left(\frac{p+n-1}{p}\right) [(n+2p-1) + \beta(n+2\alpha-1)] a_{n+p-1,m} \right\}$$

$$\leq 2\beta(p-\alpha) \left[ \prod_{i=1}^{m} a_{p-1,i} \right].$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \Sigma_{2m-1}^*(p, \alpha, \beta)$. $\qquad\square$

**Theorem 2.2.** *Let the functions $f_i(z)$ belong to the class $\Sigma S_0^*(p, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\Sigma_{m-1}^*(p, \alpha, \beta)$.*

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*Proof.* Since $f_i(z) \in \Sigma S_0^*(p, \alpha, \beta)$, we have

(2.3) $$\sum_{n=1}^{\infty} \{[(n + 2p - 1) + \beta(n + 2\alpha - 1)]\, a_{p+n-1,i}\} \leq 2\beta(p - \alpha)a_{p-1,i},$$

for $i = 1, 2, \ldots, m$. Therefore,

$$a_{p+n-1,i} \leq \left\{ \frac{2\beta(p - \alpha)}{[(n + 2p - 1) + \beta(n + 2\alpha - 1)]} \right\} a_{p-1,i}\,,$$
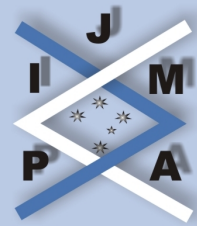
and hence

(2.4) $$a_{p+n-1,i} \leq \left( \frac{p + n - 1}{p} \right)^{-1} a_{p-1,i},$$

for every $i = 1, 2, \ldots, m$.

Using (2.4) for $i = 1, 2, \ldots, m - 1$, and (2.3) for $i = m$, we get

$$\sum_{n=1}^{\infty} \left\{ \left( \frac{p + n - 1}{p} \right)^{m-1} [(n + 2p - 1) + \beta(n + 2\alpha - 1)] \prod_{i=1}^{m} a_{p+n-1,i} \right\}$$

$$\leq \sum_{n=1}^{\infty} \left\{ \left( \frac{p + n - 1}{p} \right)^{m-1} [(n + 2p - 1) + \beta(n + 2\alpha - 1)] \right.$$

$$\left. \times \left( \left( \frac{p + n - 1}{p} \right)^{-(m-1)} \cdot \prod_{i=1}^{m-1} a_{p-1,i} \right) a_{p+n-1,m} \right\}$$

$$= \left[ \prod_{i=1}^{m-1} a_{p-1,i} \right] \sum_{n=1}^{\infty} \{[(n + 2p - 1) + \beta(n + 2\alpha - 1)]\, a_{n+p-1,m}\}$$

$$\leq 2\beta(p - \alpha) \left[ \prod_{i=1}^{m} a_{p-1,i} \right].$$

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Hence $f_1 * f_2 * \cdots * f_m(z) \in \Sigma^*_{m-1}(p, \alpha, \beta)$. $\qquad\square$

**Theorem 2.3.** *Let the functions $f_i(z)$ belong to the class $\Sigma C_0^*(p, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$, and let the functions $g_j(z)$ belong to the class $\Sigma S_0^*(p, \alpha, \beta)$ for every $j = 1, 2, \ldots, q$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\Sigma^*_{2m+q-1}(p, \alpha, \beta)$.*

*Proof.* It is sufficient to show that

$$
\sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right)^{2m+q-1} [(n+2p-1) + \beta(n+2\alpha-1)] \right.
$$

$$
\left. \times \left( \prod_{i=1}^{m} a_{p+n-1,i} \cdot \prod_{i=1}^{q} b_{p+n-1,i} \right) \right\}
$$

$$
\leq 2\beta(p-\alpha) \left( \prod_{i=1}^{m} a_{p-1,i} \prod_{i=1}^{q} b_{p-1,i} \right).
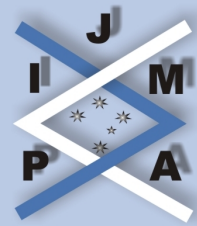$$

Since $f_i(z) \in \Sigma C_0^*(p, \alpha, \beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1, 2, \ldots, m$. Further, since $g_j(z) \in \Sigma S_0^*(p, \alpha, \beta)$, we have

$$
(2.5) \qquad \sum_{n=1}^{\infty} \left\{ [(n+2p-1) + \beta(n+2\alpha-1)] b_{p+n-1,j} \right\} \leq 2\beta(p-\alpha) b_{p-1,j},
$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$
(2.6) \qquad b_{p+n-1,j} \leq \left( \frac{p+n-1}{p} \right)^{-1} b_{p-1,j},
$$

for every $j = 1, 2, \ldots, q$.

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Using (2.2) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q-1$, and (2.5) for $j = q$, we get

$$
\sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right)^{2m+q-1} [(n+2p-1) + \beta(n+2\alpha-1)] \right.
$$
$$
\left. \times \left( \prod_{i=1}^{m} a_{p+n-1,i} \cdot \prod_{j=1}^{q} b_{p+n-1,j} \right) \right\}
$$
$$
\leq \sum_{n=1}^{\infty} \left\{ \left( \frac{p+n-1}{p} \right)^{2m+q-1} [(n+2p-1) + \beta(n+2\alpha-1)] \right.
$$
$$
\left. \times \left( \left( \frac{p+n-1}{p} \right)^{-2m} \left( \frac{p+n-1}{p} \right)^{-(q-1)} \prod_{i=1}^{m} a_{p-1,i} \prod_{j=1}^{q-1} b_{p-1,j} \right) b_{p+n-1,q} \right\}
$$
$$
= \left( \prod_{i=1}^{m} a_{p-1,i} \prod_{j=1}^{q-1} b_{p-1,j} \right) \sum_{n=1}^{\infty} \left\{ [(n+2p-1) + \beta(n+2\alpha-1)] \, b_{p+n-1,q} \right\}
$$
$$
\leq 2\beta(p-\alpha) \left( \prod_{i=1}^{m} a_{p-1,i} \prod_{j=1}^{q} b_{p-1,j} \right).
$$

Hence $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z) \in \Sigma_{2m+q-1}^{*}(p, \alpha, \beta)$.  □

We note that the required estimate can also be obtained by using (2.2) for $i = 1, 2, \ldots, m-1$, (2.6) for $j = 1, 2, \ldots, q$, and (2.1) for $i = m$.

*Remark* 1. Putting $p = 1$ in the above results, we obtain the results obtained by Mogra [10].

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