ON A GEOMETRIC INEQUALITY BY J. SÁNDOR

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Received: 02 May, 2009

Accepted: 25 September, 2009

Communicated by: L. Tóth

2000 AMS Sub. Class.: 51M16, 52A40.

Key words: Triangle, Hayashi's inequality, Hölder's inequality, Gerretsen's inequality, Euler's inequality.

Abstract: In this short note, we sharpen and generalize a geometric inequality by J. Sándor. As applications of our results, we give an alternative proof of Sándor's inequality and solve two conjectures posed by Liu.

Acknowledgements: The authors would like to thank Mr. Jian Liu and Professor J. Sándor for their careful reading and some valuable suggestions on this paper.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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vol. 10, iss. 4, art. 118, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction and Main Results

Let P be an arbitrary point P in the plane of triangle ABC. Let a, b, c be the lengths of these sides, $\triangle$ the area, s the semi-perimeter, R the circumradius and r the inradius, respectively. Denote by R_1, R_2, R_3 the distances from P to the vertices A, B, C, respectively.

The following interesting geometric inequality from 1986 is due to J. Sándor [8], a proof of this inequality can be found in the monograph [9].

Theorem 1.1. *For triangle ABC and an arbitrary point P, we have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq \frac{16}{9}\triangle^2\,. \tag{1.1}$$

Recently, J. Liu [6] also independently proved inequality (1.1).

In this short note, we sharpen and generalize inequality (1.1) and obtain the following results.

Theorem 1.2. *We have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq \frac{a^2b^2c^2}{a^2+b^2+c^2}. \tag{1.2}$$

Theorem 1.3. *If*

$$k \geq k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462,$$

then

$$(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \geq 3\left(\frac{4}{9}\sqrt{3}\triangle\right)^k\,. \tag{1.3}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Preliminary Results

Lemma 2.1 (Hayashi's inequality, see [7, pp. 297, 311]). *For any $\triangle ABC$ and an arbitrary point P, we have*

$$aR_2R_3 + bR_3R_1 + cR_1R_2 \geq abc, \tag{2.1}$$

with equality holding if and only if P is the orthocenter of the acute triangle ABC or one of the vertices of the triangle ABC.

Lemma 2.2 (see [2] and [4]). *For $\triangle ABC$, if*

$$0 \leq t \leq t_0 = \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3},$$

then we have

$$a^t + b^t + c^t \leq 3\left(\sqrt{3}R\right)^t. \tag{2.2}$$

Lemma 2.3. *Let*

$$k \geq k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462.$$

Then

$$\frac{(abc)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \geq 3\left(\frac{4}{9}\sqrt{3}\triangle\right)^k. \tag{2.3}$$

Proof. From the well known identities $abc = 4Rrs$ and $\triangle = rs$, inequality (2.3) is equivalent to

$$\frac{(4Rrs)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \geq 3\left(\frac{4}{9}\sqrt{3}rs\right)^k,$$

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journal of inequalities in pure and applied mathematics
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or

$$a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}} \le 3\left(\sqrt{3}R\right)^{\frac{k}{k-1}}. \tag{2.4}$$

It is easy to see that the function

$$f(x) = \frac{x}{x-1}$$

is strictly monotone decreasing on $(1, +\infty)$. If we let

$$t = \frac{k}{k-1} = f(k) \qquad \left(k \ge k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2}\right),$$

then

$$0 < f(k) = t \le \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3} = f(k_0),$$

and inequality (2.4) is equivalent to (2.2).

The proof of Lemma 2.3 is thus complete from Lemma 2.2. □

Lemma 2.4 ([3]). *For any $\lambda \ge 1$, we have*

$$[R-\lambda(\lambda+1)r]s^2+r[4(\lambda^2-4)R^2+(5\lambda^2+12\lambda+4)Rr+(\lambda^2+3\lambda+2)r^2] \ge 0. \tag{2.5}$$

Lemma 2.5. *In triangle ABC, we have*

$$\begin{aligned} a^9+b^9+c^9 &= 2s[s^8 - 18r(R+2r)s^6 + 18r^2(21Rr+7r^2+12R^2)s^4 \\ &\quad - 6r^3(105r^2R+240rR^2+14r^3+160R^3)s^2 + 9r^4(r+2R)(r+4R)^3]. \end{aligned}$$

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. The identity directly follows from the known identities $a + b + c = 2s$, $ab + bc + ca = s^2 + 4Rr + r^2$, $abc = 4Rrs$ and the following identity:

$$\begin{aligned}
&a^9 + b^9 + c^9 \\
&= 3a^3b^3c^3 - 45abc(ab + bc + ca)(a + b + c)^4 + 54abc(ab + bc + ca)^2(a + b + c)^2 \\
&\quad - 27a^2b^2c^2(ab + bc + ca)(a + b + c) + (a + b + c)^9 \\
&\quad - 9(ab + bc + ca)(a + b + c)^7 + 9(ab + bc + ca)^4(a + b + c) \\
&\quad - 30(ab + bc + ca)^3(a + b + c)^3 + 18a^2b^2c^2(a + b + c)^3 \\
&\quad + 27(ab + bc + ca)^2(a + b + c)^5 + 9abc(a + b + c)^6 - 9abc(ab + bc + ca)^3.
\end{aligned}$$

□

Lemma 2.6 ([5]). *If $x, y, z \geq 0$, then*

$$x + y + z + 3\sqrt[3]{xyz} \geq 2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right).$$

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issn: 1443-5756

3. Proof of the Main Result

The proof of Theorem 1.2 is easy to find from the following inequality (3.1) for $k = 2$ of the proof of Theorem 1.3. Now, we prove Theorem 1.3.

The proof of Theorem 1.3. *Hölder's inequality* and Lemma 2.1 imply for $k > 1$ that

$$\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}} \left[(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k\right]^{\frac{1}{k}} \geq aR_2R_3 + bR_3R_1 + cR_1R_2 \geq abc,$$

or

$$(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \geq \frac{(abc)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}}. \tag{3.1}$$

Combining inequality (3.1) and Lemma 2.3, we immediately see that Theorem 1.3 is true. □

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issn: 1443-5756

4. Applications

4.1. Alternative Proof of Theorem 1.1

From Theorem 1.2, in order to prove inequality (1.1), we only need to prove the following inequality:

$$\frac{a^2b^2c^2}{a^2+b^2+c^2} \geq \frac{16}{9}\triangle^2 . \tag{4.1}$$

With the known identities $abc = 4Rrs$ and $\triangle = rs$, inequality (4.1) is equivalent to

$$a^2+b^2+c^2 \leq 9R^2.$$

This is simply inequality (2.2) for $t = 2 < t_0$ in Lemma 2.2. This completes the proof of inequality (1.1).

Remark 1. The above proof of inequality (1.1) is simpler than Liu's proof [6].

4.2. Solution of Two Conjectures

In 2008, J. Liu [6] posed the following two geometric inequality conjectures, (4.2) and (4.3), involving R_1, R_2, R_3, R and r.

Conjecture 4.1. *For $\triangle ABC$ and an arbitrary point P, we have*

$$(R_1R_2)^2+(R_2R_3)^2+(R_3R_1)^2 \geq 8(R^2+2r^2)r^2, \tag{4.2}$$

and

$$(R_1R_2)^{\frac{3}{2}}+(R_2R_3)^{\frac{3}{2}}+(R_3R_1)^{\frac{3}{2}} \geq 24r^3. \tag{4.3}$$

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. First of all, from *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$s^2 \le 4R^2 + 4Rr + 3r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \ge 2r,$$

we have

$$2r^2(4R^2 + 4Rr + 3r^2 - s^2) + (R - 2r)(4R^2 + Rr + 2r^2)r \ge 0$$

$$\iff \frac{16R^2r^2s^2}{2(s^2 - 4Rr - r^2)} \ge 8(R^2 + 2r^2)r^2.$$

Using Theorem 1.2 and the known identities [7, pp.52]

$$abc = 4Rrs \qquad \text{and} \qquad a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$$

we see that inequality (4.2) holds true.

Secondly, from (3.1), in order to prove inequality (4.3), we only need to prove

$$\frac{(abc)^{\frac{3}{2}}}{[a^3 + b^3 + c^3]^{\frac{1}{2}}} \ge 24r^3. \tag{4.4}$$

With the known identities [7, pp. 52]

$$abc = 4Rrs \qquad \text{and} \qquad a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$$

inequality (4.4) is equivalent to

$$\frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^2 - 6Rr - 3r^2)]^{\frac{1}{2}}} \ge 24r^3 \tag{4.5}$$

$$\iff 18r^3(4R^2 + 4Rr + 3r^2 - s^2) + R^3(s^2 - 16Rr + 5r^2)$$
$$+ Rr(R - 2r)(16R^2 + 27Rr - 18r^2) \ge 0.$$

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issn: 1443-5756

From *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r,$$

we can conclude that inequality (4.5) holds, further, inequality (4.4) is true.

This completes the proof of Conjecture 4.1. □

Corollary 4.2. *For $\triangle ABC$ and an arbitrary point P, we have*

$$R_1^3 + R_2^3 + R_3^3 + 3R_1R_2R_3 \geq 48r^3. \tag{4.6}$$

Proof. Inequality (4.6) can directly be obtained from Lemma 2.6 and inequality (4.3). □

4.3. Sharpened Form of Above Conjectures

The inequalities (4.2) and (4.3) of Conjecture 4.1 can be sharpened as follows.

Theorem 4.3. *For $\triangle ABC$ and an arbitrary point P, we have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 8(R + r)Rr^2, \tag{4.7}$$

and

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 12Rr^2. \tag{4.8}$$

Proof. The proof of inequality (4.7) is left to the readers. Now, we prove inequality (4.8).

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issn: 1443-5756

From inequality (2.5) for $\lambda = 2$ in Lemma 2.4, the well-known *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2,$$

Euler's inequality [1, pp. 48, Theorem 5.1]

$$R \ge 2r$$

and the known identities [7, pp. 52]

$$abc = 4Rrs \quad \text{and} \quad a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$$

we obtain that

$$\begin{aligned} (4.9) \qquad & [(R-6r)s^2 + 12r^2(4R+r)] + 3r(4R^2 + 4Rr + 3r^2 - s^2) \\ & \quad + R(s^2 - 16Rr + 5r^2) + r(R-2r)(4R-3r) \ge 0 \\ & \iff \frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^2 - 6Rr - 3r^2)]^{\frac{1}{2}}} \ge 12Rr^2 \\ & \iff \frac{(abc)^{\frac{3}{2}}}{[a^3 + b^3 + c^3]^{\frac{1}{2}}} \ge 12Rr^2. \end{aligned}$$

Inequality (4.8) follows by Lemma 2.4.

Theorem 4.3 is thus proved. □

4.4. Generalization of Inequality (4.3)

Theorem 4.4. *If $k \ge \frac{9}{8}$, then*

$$(4.10) \qquad (R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \ge 3(4r^2)^k.$$

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Proof. From the monotonicity of the power mean, we only need to prove that inequality (4.10) holds for $k = \frac{9}{8}$. By using inequality (3.1), we only need to prove the following inequality

$$\frac{(abc)^{\frac{9}{8}}}{(a^9 + b^9 + c^9)^{\frac{1}{8}}} \geq 3(4r^2)^{\frac{9}{8}}. \tag{4.11}$$

From *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$s^2 \geq 16Rr - 5r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r,$$

it is obvious that

$$\begin{aligned} P = (R-2r)[&4096R^{10} + 12544R^9r + 34992R^8r^2 + 89667R^7r^3 + 218700R^6r^4 \\ &+ 516132R^5r^5 + 1189728R^4r^6 + 2493180R^3r^7 + 6018624(R-2r)Rr^8 \\ &+ 6753456r^{10} + 201204(R^2 - 4r^2)Rr^7] + 2799360r^{11} > 0, \end{aligned}$$

and

$$\begin{aligned} Q = (s^2 - 16Rr + 5r^2)\{&R^9(s^2 - 16Rr + 5r^2) \\ &+ 3R^4r(R-2r)(16R^5 + 27R^4r + 54R^3r^2 \\ &+ 108R^2r^3 + 216Rr^4 + 432r^5) + 324r^7[8(R^2 - 12r^2)^2 + 30r^2(R-2r)^2 \\ &+ 39Rr^3 + 267r^4]\} + 17496r^7(R^2 - 3Rr + 6r^2)(R^2 - 12Rr + 24r^2)^2 \\ &+ 3r^2(R-2r)\{(R-2r)[256R^9 + 864R^8r + 2457R^2r^2(R^5 - 32r^5) \\ &+ 6372R^2r^3(R^4 - 16r^4) + 15660R^2r^4(R^3 - 8r^3) + 31320R^2r^5(R^2 - 4r^2) \\ &+ 220104R^2r^6(R-2r) + 2618784(R-2r)r^8 + 51840R^2r^7 + 501120Rr^8] \\ &+ 687312r^{10}\} > 0. \end{aligned}$$

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issn: 1443-5756

Therefore, with the fundamental inequality [7, pp.1–3]

$$-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0,$$

we have

$$\begin{aligned} W &= (R^9 - 13122r^9)s^8 + 236196r^{10}(2r + R)s^6 - 236196r^{11}(7r^2 + 12R^2 + 21Rr)s^4 \\ &\quad + 78732r^{12}(105Rr^2 + 160R^3 + 240R^2r + 14r^3)s^2 \\ &\quad - 118098r^{13}(2R + r)(4R + r)^3 \\ &= 13122r^9[s^4 + 9r^3(2R + r)][-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3] \\ &\quad + r^3s^2(R - 2r)P + s^2(s^2 - 16Rr + 5r^2)Q \\ &\geq 0. \end{aligned}$$

Hence, from Lemma 2.4, we get that

$$3\left(\frac{Rs}{3r}\right)^9 - (a^9 + b^9 + c^9) = \frac{s}{6561r^9}W \geq 0, \tag{4.12}$$

or

$$3\left(\frac{Rs}{3r}\right)^9 \geq a^9 + b^9 + c^9. \tag{4.13}$$

Inequality (4.13) is simply (4.11). Thus, we complete the proof of Theorem 4.4. □

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vol. 10, iss. 4, art. 118, 2009

journal of inequalities
in pure and applied
mathematics
issn: 1443-5756

5. Two Open Problems

Finally, we pose two open problems as follows.

Open Problem 1. *For a triangle ABC and an arbitrary point P, prove or disprove*

$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \geq 72r^3. \tag{5.1}$$

Open Problem 2. *For a triangle ABC and an arbitrary point P, determine the best constant k such that the following inequality holds:*

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 12[R + k(R - 2r)]r^2. \tag{5.2}$$

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issn: 1443-5756

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issn: 1443-5756