Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 81, 2004

CORRIGENDUM TO "ON SHORT SUMS OF CERTAIN MULTIPLICATIVE FUNCTIONS"

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Received 27 August, 2003; accepted 28 April, 2004
Communicated by S.S. Dragomir

ABSTRACT. This note is a corrigendum of the main result of the paper "On short sums of certain multiplicative functions" (J. Ineq. Pure & Appl. Math., **3**(5), Art. 70 (2002)).

Key words and phrases: Corrigendum, Short sums, Integer points.

2000 *Mathematics Subject Classification.* 11N37, 11P21.

The purpose of this note is both to give a corrected statement of the main result in [1] and to provide the necessary changes to the arguments in that paper to justify this corrected statement. The result we now assert is the following :

Theorem 1. *Let $\varepsilon, c_0 > 0$ and $2 \leqslant y \leqslant c_0 x^{1/2}$ be real numbers. Let f be a multiplicative function satisfying $0 \leqslant f(n) \leqslant 1$ for any positive integer n and $f(p) = 1$ for any prime number p. We have as $x \to +\infty$:*

$$\sum_{x<n\leqslant x+y} f(n) = y\mathcal{P}(f) + O_\varepsilon\left(x^{1/15+\varepsilon} y^{2/3}\right),$$

where

$$\mathcal{P}(f) := \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{l=1}^{\infty} \frac{f(p^l)}{p^l}\right).$$

Before giving the proof, we note that if $y < x^{1/5}$, then $x^{1/15} > y^{1/3}$ so that the expression $x^{1/15+\varepsilon} y^{2/3}$ in the error term exceeds the main term (as well as the trivial bound of $y+1$ on the sum).

ISSN (electronic): 1443-5756

The author is indebted to the referee for insightful comments.
118-03

Proof. On page 5 of [1], the sum

$$S_1 := \sum_{\substack{y<d\leqslant x+y \\ d\,\text{squarefull}}} |g(d)| \left(\left[\frac{x+y}{d}\right] - \left[\frac{x}{d}\right] \right)$$

has been bounded by the sum

$$S_2 := \sum_{b\leqslant (x+y)^{1/3}} \sum_{\left(\frac{y}{b^3}\right)^{1/2}<a\leqslant\left(\frac{x+y}{b^3}\right)^{1/2}} \left(\left[\frac{(x+y)\,b^{-3}}{a^2}\right] - \left[\frac{xb^{-3}}{a^2}\right] \right)$$

by using $d = a^2b^3$ with $\mu^2(b) = 1$, and S_2 has been bounded by

$$\ll_\varepsilon x^\varepsilon \max_{1\leqslant B\leqslant (x+y)^{1/3}} \mathcal{R}\left(\frac{x}{b^3}, B, \frac{y}{B^3}\right)$$

for any (small) positive real number ε, where we defined $\mathcal{R}(f, N, \delta)$ to be the number of integer points (n, m) verifying $n \in \,]N; 2N]$ and $|f(n) - m| \leqslant \delta$. Unfortunately, the part $\sum_{b\leqslant y^{1/3}}$ of S_2 cannot be estimated by $\max_{1\leqslant B\leqslant(x+y)^{1/3}} \mathcal{R}\left(\frac{x}{b^3}, B, \frac{y}{B^3}\right)$, hence we have to proceed differently: for any positive integer r, we set

$$\tau_{(r)}(n) := \sum_{d^r|n} 1$$

and recall that

$$\tau_{(r)}(n) \ll_\varepsilon n^{\varepsilon/r}$$

for any positive integers n, r.

In S_1, $d = a^2b^3 > y$ implies $a > y^{1/5}$ or $b > y^{1/5}$. Since

$$\sum_{y^{1/5}<a\leqslant(x+y)^{1/2}} \sum_{\left(\frac{y}{a^2}\right)^{1/3}<b\leqslant\left(\frac{x+y}{a^2}\right)^{1/3}} \left(\left[\frac{(x+y)\,a^{-2}}{b^3}\right] - \left[\frac{xa^{-2}}{b^3}\right] \right) \leqslant \sum_{y^{1/5}<a\leqslant(x+y)^{1/2}} \sum_{\frac{x}{a^2}<n\leqslant\frac{x+y}{a^2}} \tau_{(3)}(n)$$

and we have the same if $b > y^{1/5}$, then

$$\begin{aligned} S_1 &\leqslant \sum_{y^{1/5}<b\leqslant(x+y)^{1/3}} \sum_{\frac{x}{b^3}<n\leqslant\frac{x+y}{b^3}} \tau_{(2)}(n) + \sum_{y^{1/5}<a\leqslant(x+y)^{1/2}} \sum_{\frac{x}{a^2}<n\leqslant\frac{x+y}{a^2}} \tau_{(3)}(n) \\ &= \sum_{y^{1/5}<b\leqslant(2y)^{1/3}} \sum_{\frac{x}{b^3}<n\leqslant\frac{x+y}{b^3}} \tau_{(2)}(n) + \sum_{(2y)^{1/3}<b\leqslant(x+y)^{1/3}} \sum_{\frac{x}{b^3}<n\leqslant\frac{x+y}{b^3}} \tau_{(2)}(n) \\ &\quad + \sum_{y^{1/5}<a\leqslant(2y)^{1/2}} \sum_{\frac{x}{a^2}<n\leqslant\frac{x+y}{a^2}} \tau_{(3)}(n) + \sum_{(2y)^{1/2}<a\leqslant(x+y)^{1/2}} \sum_{\frac{x}{a^2}<n\leqslant\frac{x+y}{a^2}} \tau_{(3)}(n) \\ &:= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \end{aligned}$$

- For Σ_1 and Σ_3 we use the trivial bound:

$$\begin{aligned} \Sigma_1 + \Sigma_3 &\ll_\varepsilon x^{\varepsilon/2} \sum_{y^{1/5}<b\leqslant(2y)^{1/3}} \left(\left[\frac{x+y}{b^3}\right] - \left[\frac{x}{b^3}\right] \right) + x^{\varepsilon/3} \sum_{y^{1/5}<a\leqslant(2y)^{1/2}} \left(\left[\frac{x+y}{a^2}\right] - \left[\frac{x}{a^2}\right] \right) \\ &\ll_\varepsilon x^{\varepsilon/2} y \left(\sum_{b>y^{1/5}} \frac{1}{b^3} + \sum_{a>y^{1/5}} \frac{1}{a^2} \right) \ll_\varepsilon y^{4/5} x^{\varepsilon/2}. \end{aligned}$$

- For Σ_2, we use the method of [1] to get
$$\Sigma_2 \ll_\varepsilon x^\varepsilon \left(x^{1/6} + y^{1/3}\right).$$
- For Σ_4, if we suppose $y \leqslant c_0 x^{1/2}$ (where $c_0 > 0$ is sufficiently small), we have using Lemmas 2.1 and 2.2 of [1]:
$$\begin{aligned}\Sigma_4 &\ll_\varepsilon x^\varepsilon \left\{ \max_{(2y)^{1/2} < A \leqslant c_0^{-1} y} \mathcal{R}\left(\frac{x}{a^2}, A, \frac{y}{A^2}\right) + \max_{c_0^{-1} y < A \leqslant x^{1/2}} \mathcal{R}\left(\frac{x}{a^2}, A, \frac{y}{A^2}\right) \right\} \\ &\ll_\varepsilon x^\varepsilon \left((xy)^{1/6} + x^{1/5} + x^{1/15} y^{2/3}\right).\end{aligned}$$

Hence we finally have
$$\begin{aligned}S_1 &\ll_\varepsilon x^\varepsilon \left(x^{1/15} y^{2/3} + y^{4/5} + (xy)^{1/6} + x^{1/5} + x^{1/6} + y^{1/3}\right) \\ &\ll_\varepsilon x^\varepsilon \left(x^{1/15} y^{2/3} + y^{4/5}\right)\end{aligned}$$
if $y \geqslant x^{1/5}$. Note that $y^{4/5} \ll x^{1/15} y^{2/3}$ if $y \leqslant c_0 x^{1/2}$ and that
$$\left| \sum_{x < n \leqslant x+y} f(n) - y\mathcal{P}(f) \right| \ll y \ll x^{1/15} y^{2/3}$$
if $y < x^{1/5}$. This concludes the proof of the theorem. □

References

[1] O. BORDELLÈS, On short sums of certain multiplicative functions, *J. Inequal. Pure and Appl. Math.*, **3**(5) (2002), Art. 70. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=222`]