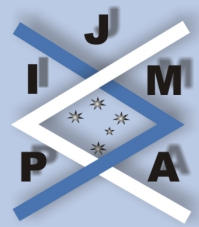


λ -CENTRAL BMO ESTIMATES FOR COMMUTATORS OF N -DIMENSIONAL HARDY OPERATORS

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- Abstract:* This paper gives the λ -central BMO estimates for commutators of n -dimensional Hardy operators on central Morrey spaces.
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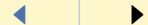
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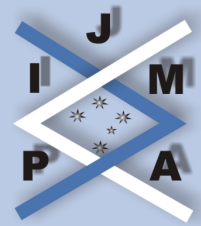
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1. Introduction and Main Results

Let f be a locally integrable function on \mathbb{R}^n . The n -dimensional Hardy operators are defined by

$$\mathcal{H}f(x) := \frac{1}{|x|^n} \int_{|t| \leq |x|} f(t) dt, \quad \mathcal{H}^*f(x) := \int_{|t| > |x|} \frac{f(t)}{|t|^n} dt, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

In [4], Christ and Grafakos obtained results for the boundedness of \mathcal{H} on $L^p(\mathbb{R}^n)$ spaces. They also found the exact operator norms of \mathcal{H} on $L^p(\mathbb{R}^n)$ spaces, where $1 < p < \infty$.

It is easy to see that \mathcal{H} and \mathcal{H}^* satisfy

$$(1.1) \quad \int_{\mathbb{R}^n} g(x) \mathcal{H}f(x) dx = \int_{\mathbb{R}^n} f(x) \mathcal{H}^*g(x) dx.$$

We have

$$|\mathcal{H}f(x)| \leq C_n Mf(x),$$

where M is the Hardy-Littlewood maximal operator which is defined by

$$(1.2) \quad Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(t)| dt,$$

where the supremum is taken over all balls containing x .

Recently, Fu et al. [2] gave the definition of commutators of n -dimensional Hardy operators.

Definition 1.1. Let b be a locally integrable function on \mathbb{R}^n . We define the commutators of n -dimensional Hardy operators as follows:

$$\mathcal{H}_b f := b\mathcal{H}f - \mathcal{H}(fb), \quad \mathcal{H}_b^* f := b\mathcal{H}^*f - \mathcal{H}^*(fb).$$



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In [2], Fu et al. gave the central BMO estimates for commutators of n -dimensional Hardy operators. In 2000, Alvarez, Guzmán-Partida and Lakey [1] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ -central bounded mean oscillation spaces and central Morrey spaces, respectively.

Definition 1.2 (λ -central BMO space). Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < \frac{1}{n}$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the λ -central bounded mean oscillation space $CMO^{q,\lambda}(\mathbb{R}^n)$ if

$$(1.3) \quad \|f\|_{CMO^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_{B(0,R)}|^q dx \right)^{\frac{1}{q}} < \infty.$$

Remark 1. If two functions which differ by a constant are regarded as a function in the space $CMO^{q,\lambda}(\mathbb{R}^n)$, then $CMO^{q,\lambda}(\mathbb{R}^n)$ becomes a Banach space. Apparently, (1.3) is equivalent to the following condition (see [1]):

$$\sup_{R>0} \inf_{c \in \mathbb{C}} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x) - c|^q dx \right)^{\frac{1}{q}} < \infty.$$

Definition 1.3 (Central Morrey spaces, see [1]). Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < 0$. The central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined by

$$(1.4) \quad \|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x)|^q dx \right)^{\frac{1}{q}} < \infty.$$

Remark 2. It follows from (1.3) and (1.4) that $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is a Banach space continuously included in $CMO^{q,\lambda}(\mathbb{R}^n)$.



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Inspired by [2], [3] and [5], we will establish the λ -central BMO estimates for commutators of n -dimensional Hardy operators on central Morrey spaces.

Theorem 1.4. *Let \mathcal{H}_b be defined as above. Suppose $1 < p_1 < \infty$, $p'_1 < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0$, $0 \leq \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in CMO^{p_2, \lambda_2}(\mathbb{R}^n)$, then the commutator \mathcal{H}_b is bounded from $\dot{B}^{p_1, \lambda}(\mathbb{R}^n)$ to $\dot{B}^{q, \lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q, \lambda}(\mathbb{R}^n)} \leq C \|b\|_{CMO^{p_2, \lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.4. We can obtain the central BMO estimates for commutators of n -dimensional Hardy operators, \mathcal{H}_b , on central Morrey spaces.

Corollary 1.5. *Let \mathcal{H}_b be defined as above. Suppose $1 < p_1 < \infty$, $p'_1 < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ and $-\frac{1}{q} < \lambda < 0$. If $b \in CMO^{p_2}(\mathbb{R}^n)$, then the commutator \mathcal{H}_b is bounded from $\dot{B}^{p_1, \lambda}(\mathbb{R}^n)$ to $\dot{B}^{q, \lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q, \lambda}(\mathbb{R}^n)} \leq C \|b\|_{CMO^{p_2}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p_1, \lambda}(\mathbb{R}^n)}.$$

Similar to Theorem 1.4, we have:

Theorem 1.6. *Let \mathcal{H}_b^* be defined as above. Suppose $1 < p_1 < \infty$, $p'_1 < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0$, $0 \leq \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in CMO^{p_2, \lambda_2}(\mathbb{R}^n)$, then the commutator \mathcal{H}_b^* is bounded from $\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)$ to $\dot{B}^{q, \lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q, \lambda}(\mathbb{R}^n)} \leq C \|b\|_{CMO^{p_2, \lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.6. We can get the central BMO estimates for commutators of n -dimensional Hardy operators, \mathcal{H}_b^* , on central Morrey spaces.



Corollary 1.7. Let \mathcal{H}_b^* be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ and $-\frac{1}{q} < \lambda < 0$. If $b \in \dot{C}MO^{p_2}(\mathbb{R}^n)$, then the commutator \mathcal{H}_b^* is bounded from $\dot{B}^{p_1, \lambda}(\mathbb{R}^n)$ to $\dot{B}^{q, \lambda}(\mathbb{R}^n)$ and satisfies the following inequality:

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q, \lambda}(\mathbb{R}^n)} \leq C \|b\|_{\dot{C}MO^{p_2}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p_1, \lambda}(\mathbb{R}^n)}.$$

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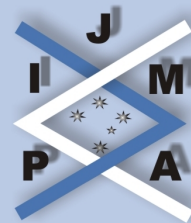
2. Proofs of Theorems

Proof of Theorem 1.4. Let f be a function in $\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)$. For fixed $R > 0$, denote $B(0, R)$ by B . Write

$$\begin{aligned} & \left(\frac{1}{|B|} \int_B |\mathcal{H}_b f(x)|^q dx \right)^{\frac{1}{q}} \\ &= \left(\frac{1}{|B|} \int_B \left| \frac{1}{|x|^n} \int_{B(0, |x|)} f(y)(b(x) - b(y)) dy \right|^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{|B|} \int_B \left| \frac{1}{|x|^n} \int_{B(0, |x|)} f(y)(b(x) - b_B) dy \right|^q dx \right)^{\frac{1}{q}} \\ &\quad + \left(\frac{1}{|B|} \int_B \left| \frac{1}{|x|^n} \int_{B(0, |x|)} f(y)(b(y) - b_B) dy \right|^q dx \right)^{\frac{1}{q}} \\ &:= I + J. \end{aligned}$$

For $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, by Hölder's inequality and the boundedness of \mathcal{H} from L^{p_1} to L^{p_1} , we have

$$\begin{aligned} I &\leq |B|^{-\frac{1}{q}} \left(\int_B |b(x) - b_B|^{p_2} dx \right)^{\frac{1}{p_2}} \left(\int_B |\mathcal{H}(f\chi_B)(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ &\leq C|B|^{-\frac{1}{q}} \|b\|_{\dot{C}MO^{p_2, \lambda_2}(\mathbb{R}^n)} |B|^{\frac{1}{p_2} + \lambda_2} \left(\int_B |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ &= C|B|^\lambda \|b\|_{\dot{C}MO^{p_2, \lambda_2}(\mathbb{R}^n)} \left(\frac{1}{|B|^{1+p_1\lambda_1}} \int_B |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ &\leq C|B|^\lambda \|b\|_{\dot{C}MO^{p_2, \lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}. \end{aligned}$$



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For J , we have

$$\begin{aligned}
 J^q &= \frac{1}{|B|} \int_B \left| \frac{1}{|x|^n} \int_{B(0, |x|)} f(y)(b(y) - b_B) dy \right|^q dx \\
 &= \frac{1}{|B|} \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^n} \int_{B(0, |x|)} f(y)(b(y) - b_B) dy \right|^q dx \\
 &\leq \frac{C}{|B|} \sum_{k=-\infty}^0 \frac{1}{|2^k B|^q} \int_{2^k B \setminus 2^{k-1} B} \left| \sum_{i=-\infty}^k \int_{2^i B \setminus 2^{i-1} B} f(y)(b(y) - b_B) dy \right|^q dx \\
 &\leq \frac{C}{|B|} \sum_{k=-\infty}^0 \frac{1}{|2^k B|^q} \int_{2^k B \setminus 2^{k-1} B} \left| \sum_{i=-\infty}^k \int_{2^i B \setminus 2^{i-1} B} f(y)(b(y) - b_{2^i B}) dy \right|^q dx \\
 &\quad + \frac{C}{|B|} \sum_{k=-\infty}^0 \frac{1}{|2^k B|^q} \int_{2^k B \setminus 2^{k-1} B} \left| \sum_{i=-\infty}^k \int_{2^i B \setminus 2^{i-1} B} f(y)(b_{2^i B} - b_B) dy \right|^q dx \\
 &:= J_1 + J_2
 \end{aligned}$$

By Hölder's inequality ($\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}$), we have

$$\begin{aligned}
 J_1 &\leq \frac{C}{|B|} \sum_{k=-\infty}^0 \frac{|2^k B|}{|2^k B|^q} \left\{ \sum_{i=-\infty}^k |2^i B|^{\frac{1}{q'}} \left(\int_{2^i B} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} \right. \\
 &\quad \left. \times \left(\int_{2^i B} |b(y) - b_{2^i B}|^{p_2} dy \right)^{\frac{1}{p_2}} \right\}^q \\
 &\leq \frac{C}{|B|} \|b\|_{CMOP_{p_2, \lambda_2}(\mathbb{R}^n)}^q \|f\|_{BP_{p_1, \lambda_1}(\mathbb{R}^n)}^q \sum_{k=-\infty}^0 \frac{|2^k B|}{|2^k B|^q} \left\{ \sum_{i=-\infty}^k |2^i B|^{\lambda+1} \right\}^q
 \end{aligned}$$

$$\leq C|B|^{q\lambda} \|b\|_{C\dot{M}O^{p_2, \lambda_2}(\mathbb{R}^n)}^q \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}^q.$$

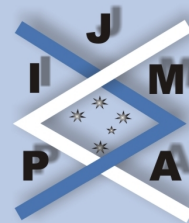
To estimate J_2 , the following fact is applied.

For $\lambda_2 \geq 0$,

$$\begin{aligned} |b_{2^i B} - b_B| &\leq \sum_{j=i}^{-1} |b_{2^{j+1} B} - b_{2^j B}| \\ &\leq \sum_{j=i}^{-1} \frac{1}{|2^j B|} \int_{2^j B} |b(y) - b_{2^{j+1} B}| dy \\ &\leq C \sum_{j=i}^{-1} \left(\frac{1}{|2^{j+1} B|} \int_{2^{j+1} B} |b(y) - b_{2^{j+1} B}|^{p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq C \|b\|_{C\dot{M}O^{p_2, \lambda_2}(\mathbb{R}^n)} |B|^{\lambda_2} \sum_{j=i}^{-1} 2^{(j+1)n\lambda_2} \\ &\leq C \|b\|_{C\dot{M}O^{p_2, \lambda_2}(\mathbb{R}^n)} |i| |B|^{\lambda_2}. \end{aligned}$$

By Hölder's inequality ($\frac{1}{p_1} + \frac{1}{p_1'} = 1$), we have

$$\begin{aligned} J_2 &= \frac{C}{|B|} \sum_{k=-\infty}^0 \frac{1}{|2^k B|^q} \int_{2^k B \setminus 2^{k-1} B} \left| \sum_{i=-\infty}^k \int_{2^i B \setminus 2^{i-1} B} f(y)(b_{2^i B} - b_B) dy \right|^q dx \\ &\leq \frac{C}{|B|} \|b\|_{C\dot{M}O^{p_2, \lambda_2}(\mathbb{R}^n)}^q \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}^q \sum_{k=-\infty}^0 \frac{|2^k B| |B|^{q\lambda_2}}{|2^k B|^q} \left\{ \sum_{i=-\infty}^k |i| |2^i B|^{\lambda_1+1} \right\}^q \\ &\leq \frac{C}{|B|} \|b\|_{C\dot{M}O^{p_2, \lambda_2}(\mathbb{R}^n)}^q \|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}^q \sum_{k=-\infty}^0 \frac{|2^k B| |B|^{q\lambda_2} |k|^q |2^k B|^{(\lambda_1+1)q}}{|2^k B|^q} \end{aligned}$$



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$$\leq C|B|^{q\lambda}\|b\|_{CMO^{p_2, \lambda_2}(\mathbb{R}^n)}^q\|f\|_{\dot{B}^{p_1, \lambda_1}(\mathbb{R}^n)}^q.$$

Combining the estimates of I , J_1 and J_2 , we get the required estimate for Theorem 1.4. □

Proof of Theorem 1.6. We omit the details here. □



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