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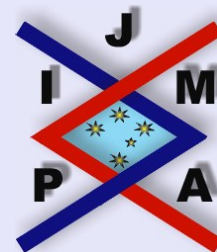
EQUATIONS AND INEQUALITIES INVOLVING $v_p(n!)$

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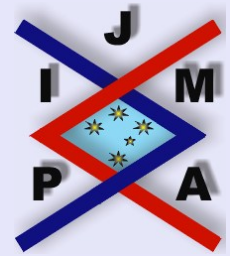


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Abstract

In this paper we study $v_p(n!)$, the greatest power of prime p in factorization of $n!$. We find some lower and upper bounds for $v_p(n!)$, and we show that $v_p(n!) = \frac{n}{p-1} + O(\ln n)$. By using the afore mentioned bounds, we study the equation $v_p(n!) = v$ for a fixed positive integer v . Also, we study the triangle inequality about $v_p(n!)$, and show that the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ holds for primes $p < q$ and sufficiently large values of n .

2000 Mathematics Subject Classification: 05A10, 11A41, 26D15, 26D20.

Key words: Factorial function, Prime number, Inequality.

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Dedicated to Professor Bahman Mehri on the occasion of his 70th birthday

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1. Introduction

As we know, for every $n \in \mathbb{N}$, $n! = 1 \times 2 \times 3 \times \cdots \times n$. Let $v_p(n!)$ be the highest power of prime p in factorization of $n!$ to prime numbers. It is well-known that (see [3] or [5])

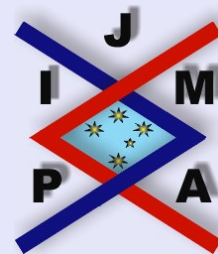
$$(1.1) \quad v_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k} \right] = \sum_{k=1}^{\left[\frac{\ln n}{\ln p} \right]} \left[\frac{n}{p^k} \right],$$

in which $[x]$ is the largest integer less than or equal to x . An elementary problem about $n!$ is finding the number of zeros at the end of it, in which clearly its answer is $v_5(n!)$. The inverse of this problem is very nice; for example finding values of n in which $n!$ terminates in 37 zeros [3], and generally finding values of n such that $v_p(n!) = v$. We show that if $v_p(n!) = v$ has a solution then it has exactly p solutions. For doing these, we need some properties of $[x]$, such as

$$(1.2) \quad [x] + [y] \leq [x + y] \quad (x, y \in \mathbb{R}),$$

and

$$(1.3) \quad \left[\frac{x}{n} \right] = \left[\frac{[x]}{n} \right] \quad (x \in \mathbb{R}, n \in \mathbb{N}).$$



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2. Estimating $v_p(n!)$

Theorem 2.1. For every $n \in \mathbb{N}$ and prime p , such that $p \leq n$, we have:

$$(2.1) \quad \frac{n-p}{p-1} - \frac{\ln n}{\ln p} < v_p(n!) \leq \frac{n-1}{p-1}.$$

Proof. According to the relation (1.1), we have $v_p(n!) = \sum_{k=1}^m \left[\frac{n}{p^k} \right]$ in which $m = \left[\frac{\ln n}{\ln p} \right]$, and since $x-1 < [x] \leq x$, we obtain

$$n \sum_{k=1}^m \frac{1}{p^k} - m < v_p(n!) \leq n \sum_{k=1}^m \frac{1}{p^k},$$

considering $\sum_{k=1}^m \frac{1}{p^k} = \frac{1 - \frac{1}{p^{m+1}}}{p-1}$, we obtain

$$\frac{n}{p-1} \left(1 - \frac{1}{p^{m+1}} \right) - m < v_p(n!) \leq \frac{n}{p-1} \left(1 - \frac{1}{p^{m+1}} \right),$$

and combining this inequality with $\frac{\ln n}{\ln p} - 1 < m \leq \frac{\ln n}{\ln p}$ completes the proof. \square

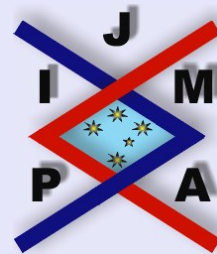
Corollary 2.2. For every $n \in \mathbb{N}$ and prime p , such that $p \leq n$, we have:

$$v_p(n!) = \frac{n}{p-1} + O(\ln n).$$

Proof. By using (2.1), we have

$$0 < \frac{\frac{n}{p-1} - v_p(n!)}{\ln n} < \frac{1}{\ln p} + O\left(\frac{1}{\ln n}\right),$$

and this yields the result. \square



Note that the above corollary asserts that $n!$ ends approximately in $\frac{n}{4}$ zeros [1].

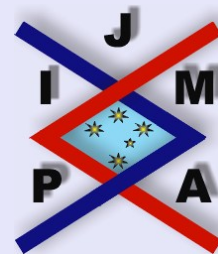
Corollary 2.3. For every $n \in \mathbb{N}$ and prime p , such that $p \leq n$, and for all $a \in (0, \infty)$ we have:

$$(2.2) \quad \frac{n-p}{p-1} - \frac{1}{\ln p} \left(\frac{n}{a} + \ln a - 1 \right) < v_p(n!).$$

Proof. Consider the function $f(x) = \ln x$. Since, $f''(x) = -\frac{1}{x^2}$, $\ln x$ is a concave function and so, for every $a \in (0, +\infty)$ we have

$$\ln x \leq \ln a + \frac{1}{a}(x - a),$$

combining this with the left hand side of (2.1) completes the proof. □



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3. Study of the Equation $v_p(n!) = v$

Suppose $v \in \mathbb{N}$ is given. We are interested in finding the values of n such that in factorization of $n!$, the highest power of p , is equal to v . First, we find some lower and upper bounds for these n 's.

Lemma 3.1. *Suppose $v \in \mathbb{N}$ and p is a prime and $v_p(n!) = v$, then we have*

$$(3.1) \quad 1 + (p-1)v \leq n < \frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{1}{p-1} - \frac{1}{(1+(p-1)v)\ln p}}.$$

Proof. For proving the left hand side of (3.1), use right hand side of (2.1) with the assumption $v_p(n!) = v$, and for proving the right hand side of (3.1), use (2.2) with $a = 1 + (p-1)v$. \square

Lemma 3.1 suggests an interval for the solution of $v_p(n!) = v$. In the next lemma we show that it is sufficient for one to check only multiples of p in above interval.

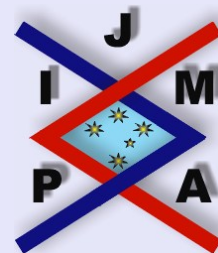
Lemma 3.2. *Suppose $m \in \mathbb{N}$ and p is a prime, then we have*

$$(3.2) \quad v_p((pm+p)!) - v_p((pm)!) \geq 1.$$

Proof. By using (1.1) and (1.2) we have

$$\begin{aligned} v_p((pm+p)!) &= \sum_{k=1}^{\infty} \left[\frac{pm+p}{p^k} \right] \geq \sum_{k=1}^{\infty} \left[\frac{pm}{p^k} \right] + \sum_{k=1}^{\infty} \left[\frac{p}{p^k} \right] \\ &= 1 + v_p((pm)!), \end{aligned}$$

and this completes the proof. \square



In the next lemma, we show that if $v_p(n!) = v$ has a solution, then it has exactly p solutions. In fact, the next lemma asserts that if $v_p((mp)!) = v$ holds, then for all $0 \leq r \leq p - 1$, $v_p((mp + r)!) = v$ also holds.

Lemma 3.3. *Suppose $m \in \mathbb{N}$ and p is a prime, then we have*

$$(3.3) \quad v_p((m + 1)!) \geq v_p(m!),$$

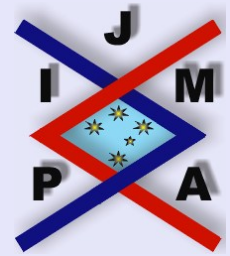
and

$$(3.4) \quad v_p((pm + p - 1)!) = v_p((pm)!).$$

Proof. For proving (3.3), use (1.1) and (1.2) as follows

$$\begin{aligned} v_p((m + 1)!) &= \sum_{k=1}^{\infty} \left[\frac{m + 1}{p^k} \right] \\ &\geq \sum_{k=1}^{\infty} \left[\frac{m}{p^k} \right] + \sum_{k=1}^{\infty} \left[\frac{1}{p^k} \right] \\ &= \sum_{k=1}^{\infty} \left[\frac{m}{p^k} \right] = v_p(m!). \end{aligned}$$

For proving (3.4), it is enough to show that for all $k \in \mathbb{N}$, $\left[\frac{pm+p-1}{p^k} \right] = \left[\frac{pm}{p^k} \right]$ and we do this by induction on k ; for $k = 1$, clearly $\left[\frac{pm+p-1}{p} \right] = \left[\frac{pm}{p} \right]$. Now,



by using (1.3) we have

$$\begin{aligned} \left[\frac{pm + p - 1}{p^{k+1}} \right] &= \left[\frac{\frac{pm+p-1}{p^k}}{p} \right] = \left[\frac{\left[\frac{pm+p-1}{p^k} \right]}{p} \right] = \left[\frac{\left[\frac{pm}{p^k} \right]}{p} \right] \\ &= \left[\frac{\frac{pm}{p^k}}{p} \right] = \left[\frac{pm}{p^{k+1}} \right]. \end{aligned}$$

This completes the proof. \square

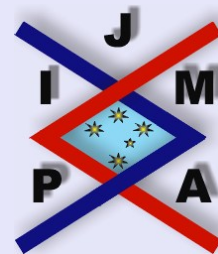
So, we have proved that

Theorem 3.4. *Suppose $v \in \mathbb{N}$ and p is a prime. For solving the equation $v_p(n!) = v$, it is sufficient to check the values $n = mp$, in which $m \in \mathbb{N}$ and*

$$(3.5) \quad \left[\frac{1 + (p-1)v}{p} \right] \leq m \leq \left[\frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{p}{p-1} - \frac{p}{(1+(p-1)v)\ln p}} \right].$$

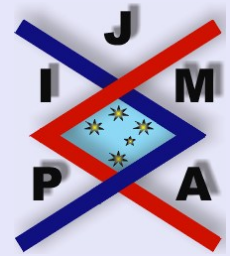
Also, if $n = mp$ is a solution of $v_p(n!) = v$, then it has exactly p solutions $n = mp + r$, in which $0 \leq r \leq p - 1$.

Note and Problem 1. *As we see, there is no guarantee of the existence of a solution for $v_p(n!) = v$. In fact we need to show that $\{v_p(n!) | n \in \mathbb{N}\} = \mathbb{N}$; however, computational observations suggest that $n = p \left\| \frac{1+(p-1)v}{p} \right\|$ usually is a solution, such that $\|x\|$ is the nearest integer to x , but we cannot prove it.*



Note and Problem 2. *Other problems can lead us to other equations involving $v_p(n!)$; for example, suppose $n, v \in \mathbb{N}$ given, find the value of prime p such that $v_p(n!) = v$.*

Or, suppose p and q are primes and $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ is a prime value function, for which n 's do we have $v_p(n!) + v_q(n!) = v_{f(p,q)}(n!)$? And many other problems!



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4. Triangle Inequality Concerning $v_p(n!)$

In this section we are going to compare $v_p((m+n)!)$ and $v_p(m!) + v_p(n!)$.

Theorem 4.1. For every $m, n \in \mathbb{N}$ and prime p , such that $p \leq \min\{m, n\}$, we have

$$(4.1) \quad v_p((m+n)!) \geq v_p(m!) + v_p(n!),$$

and

$$(4.2) \quad v_p((m+n)!) - v_p(m!) - v_p(n!) = O(\ln(mn)).$$

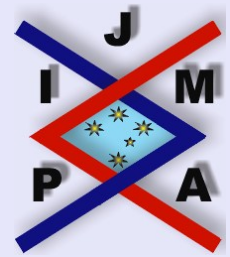
Proof. By using (1.1) and (1.2), we have

$$\begin{aligned} v_p((m+n)!) &= \sum_{k=1}^{\infty} \left[\frac{m+n}{p^k} \right] \\ &\geq \sum_{k=1}^{\infty} \left[\frac{m}{p^k} \right] + \sum_{k=1}^{\infty} \left[\frac{n}{p^k} \right] \\ &= v_p(m!) + v_p(n!). \end{aligned}$$

Also, by using (2.1) and (4.1) we obtain

$$\begin{aligned} 0 &\leq v_p((m+n)!) - v_p(m!) - v_p(n!) \\ &< \frac{2p-1}{p-1} + \frac{\ln(mn)}{\ln p} \leq 3 + \frac{\ln(mn)}{\ln 2}, \end{aligned}$$

this completes the proof. \square



More generally, if $n_1, n_2, \dots, n_t \in \mathbb{N}$ and p is a prime, in which $p \leq \min\{n_1, n_2, \dots, n_t\}$, by using an extension of (1.2), we obtain

$$v_p \left(\left(\sum_{k=1}^t n_k \right)! \right) \geq \sum_{k=1}^t v_p(n_k!),$$

and by using this inequality and (2.1), we obtain

$$\begin{aligned} 0 &\leq v_p \left(\left(\sum_{k=1}^t n_k \right)! \right) - \sum_{k=1}^t v_p(n_k!) \\ &< \frac{tp-1}{p-1} + \frac{\ln(n_1 n_2 \cdots n_t)}{\ln p} \\ &\leq 2t-1 + \frac{\ln(n_1 n_2 \cdots n_t)}{\ln 2}, \end{aligned}$$

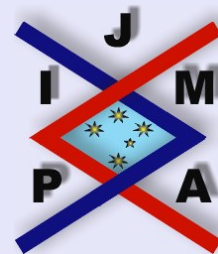
and consequently we have

$$v_p \left(\left(\sum_{k=1}^t n_k \right)! \right) - \sum_{k=1}^t v_p(n_k!) = O(\ln(n_1 n_2 \cdots n_t)).$$

Note and Problem 3. Suppose $f : \mathbb{N}^t \rightarrow \mathbb{N}$ is a function and p is a prime. For which $n_1, n_2, \dots, n_t \in \mathbb{N}$, do we have

$$v_p((f(n_1, n_2, \dots, n_t))!) \geq f(v_p(n_1!), v_p(n_2!), \dots, v_p(n_t!))?$$

Also, we can consider the above question in other view points.



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5. The Inequality $p^{v_p(n!)} > q^{v_q(n!)}$

Suppose p and q are primes and $p < q$. Since $v_p(n!) \geq v_q(n!)$, comparing $p^{v_p(n!)}$ and $q^{v_q(n!)}$ becomes a nice problem. In [2], by using elementary properties about $[x]$, the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ was considered for some special cases. In addition, it was shown that $2^{v_2(n!)} > 3^{v_3(n!)}$ holds for all $n \geq 4$. In this section we study $p^{v_p(n!)} > q^{v_q(n!)}$ in the more general case and also reprove $2^{v_2(n!)} > 3^{v_3(n!)}$.

Lemma 5.1. *Suppose p and q are primes and $p < q$, then*

$$p^{q-1} > q^{p-1}.$$

Proof. Consider the function

$$f(x) = x^{\frac{1}{x-1}} \quad (x \geq 2).$$

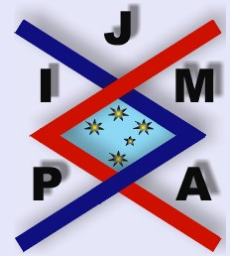
A simple calculation yields that for $x \geq 2$ we have

$$f'(x) = -\frac{x^{\frac{2-x}{x-1}}(x \ln x - x + 1)}{(x-1)^2} < 0,$$

so, f is strictly decreasing and $f(p) > f(q)$. This completes the proof. \square

Theorem 5.2. *Suppose p and q are primes and $p < q$, then for sufficiently large n 's we have*

$$(5.1) \quad p^{v_p(n!)} > q^{v_q(n!)}.$$



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Proof. Since $p < q$, Lemma 5.1 yields that $\frac{p^{q-1}}{q^{p-1}} > 1$ and so, there exists $N \in \mathbb{N}$ such that for $n > N$ we have

$$\left(\frac{p^{q-1}}{q^{p-1}}\right)^n \geq \frac{p^{p(q-1)}}{q^{p-1}} n^{(p-1)(q-1)}.$$

Thus,

$$\frac{p^{n(q-1)}}{n^{(p-1)(q-1)} p^{p(q-1)}} \geq \frac{q^{n(p-1)}}{q^{p-1}},$$

and therefore,

$$\frac{p^{\frac{n}{p-1}}}{np^{\frac{p}{p-1}}} \geq \frac{q^{\frac{n}{q-1}}}{q^{\frac{1}{q-1}}}.$$

So, we obtain

$$p^{\frac{n-p}{p-1} - \frac{\ln n}{\ln p}} \geq q^{\frac{n-1}{q-1}},$$

and considering this inequality with (2.1), completes the proof. \square

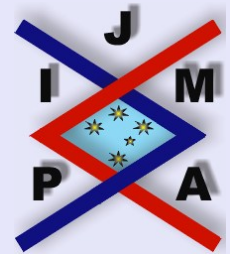
Corollary 5.3. For $n = 2$ and $n \geq 4$ we have

$$(5.2) \quad 2^{v_2(n!)} > 3^{v_3(n!)}.$$

Proof. It is easy to see that for $n \geq 30$ we have

$$\left(\frac{4}{3}\right)^n \geq \frac{16}{3} n^2,$$

and by Theorem 5.2, we yield (5.2) for $n \geq 30$. For $n = 2$ and $4 \leq n < 30$ check it using a computer. \square



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A Computational Note. In Theorem 5.2, the relation (5.1) holds for $n > N$ (see its proof). We can check (5.1) for $n \leq N$ at most by checking the following number of cases:

$$R(N) := \# \{(p, q, n) \mid p, q \in \mathbb{P}, n = 3, 4, \dots, N, \text{ and } p < q \leq N\},$$

in which \mathbb{P} is the set of all primes. If, $\pi(x) =$ The number of primes $\leq x$, then we have

$$R(N) = \sum_{n=3}^N \# \{(p, q) \mid p, q \in \mathbb{P}, \text{ and } p < q \leq n\} = \frac{1}{2} \sum_{n=3}^N \pi(n)(\pi(n) - 1).$$

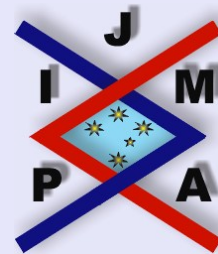
But, clearly $\pi(n) < n$ and this yields that

$$R(N) < \frac{N^3}{6}.$$

Of course, we have other bounds for $\pi(n)$ sharper than n such as [4]

$$\pi(n) \leq \frac{n}{\ln n} \left(1 + \frac{1}{\ln n} + \frac{2.25}{\ln^2 n} \right) \quad (n \geq 355991),$$

and by using this bound we can find sharper bounds for $R(N)$.



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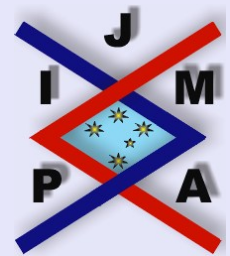
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