

p -VALENT MEROMORPHIC FUNCTIONS WITH ALTERNATING COEFFICIENTS BASED ON INTEGRAL OPERATOR

SH. NAJAFZADEH

Department of Mathematics, Faculty of Science
Maragheh University, Maragheh, Iran

EMail: najafzadeh1234@yahoo.ie

A. EBADIAN AND S. SHAMS

Department of Mathematics, Faculty of Science
Urmia University, Urmia, Iran

EMail: a.ebadian@mail.urmia.ac.ir

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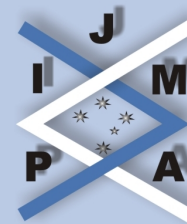
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Abstract: By using a linear operator, a subclass of meromorphically p -valent functions with alternating coefficients is introduced. Some important properties of this class such as coefficient bounds, distortion bounds, etc. are found.



p -valent Meromorphic
Functions

Sh. Najafzadeh, A. Ebadian
and S. Shams

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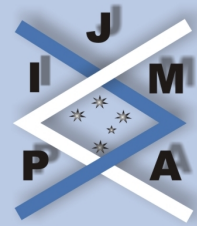
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1. Introduction

Let Σ_p be the class of functions of the form

$$(1.1) \quad f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n, \quad A \geq 0$$

that are regular in the punctured disk $\Delta^* = \{z : 0 < |z| < 1\}$ and σ_p be the subclass of Σ_p consisting of functions with alternating coefficients of the type

$$(1.2) \quad f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n, \quad a_n \geq 0, \quad A \geq 0.$$

Let

$$(1.3) \quad \Sigma_p^*(\beta) = \left\{ f \in \Sigma_p : \operatorname{Re} \left(\frac{z[\mathcal{J}(f(z))]' }{\mathcal{J}(f(z))} \right) < -\beta, 0 \leq \beta < p \right\}$$

and let $\sigma_p^*(\beta) = \Sigma_p^*(\beta) \cap \sigma_p$ where

$$(1.4) \quad \mathcal{J}(f(z)) = (\gamma - p + 1) \int_0^1 (u^\gamma) f(uz) du, \quad p < \gamma$$

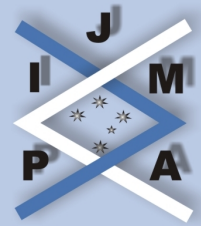
is a linear operator.

With a simple calculation we obtain

$$(1.5) \quad \mathcal{J}(f(z)) = \begin{cases} Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n, & f(z) \in \sigma_p; \\ Az^{-p} + \sum_{n=p}^{\infty} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n, & f(z) \in \Sigma_p. \end{cases}$$

For more details about meromorphic p -valent functions, we can see the recent works of many authors in [1], [2], [3].

Also, Uralegaddi and Ganigi [4] worked on meromorphic univalent functions with alternating coefficients.



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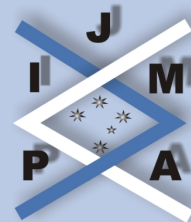
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2. Coefficient Estimates

Theorem 2.1. *Let*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \Sigma_p.$$

If

$$(2.1) \quad \sum_{n=p}^{\infty} (n + \beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n| \leq A(p - \beta),$$

then $f(z) \in \Sigma_p^*(\beta)$.

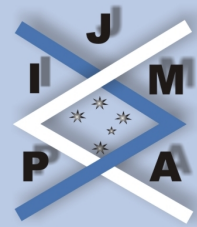
Proof. It is sufficient to show that

$$M = \left| \frac{\frac{z[\mathcal{J}f(z)]'}{\mathcal{J}f(z)} + p}{\frac{z[\mathcal{J}f(z)]'}{\mathcal{J}f(z)} - (p - 2\beta)} \right| < 1 \quad \text{for } |z| < 1.$$

However, by (1.5)

$$M = \left| \frac{-pAz^{-p} + \sum_{n=p}^{\infty} n \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n + pAz^{-p} + \sum_{n=p}^{\infty} p \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n}{-pAz^{-p} + \sum_{n=p}^{\infty} n \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n - (p - 2\beta)Az^{-p} - \sum_{n=p}^{\infty} (p - 2\beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n} \right|$$

$$\leq \frac{\sum_{n=p}^{\infty} \left[(n + p) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) \right] |a_n|}{2A(p - \beta) - \sum_{n=p}^{\infty} (n - p + 2\beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n|}.$$



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The last expression is less than or equal to 1 provided

$$\sum_{n=p}^{\infty} \left[(n+p) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) \right] |a_n| \leq 2A(p-\beta) - \sum_{n=p}^{\infty} (n-p+2\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) |a_n|,$$

which is equivalent to

$$\sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) |a_n| \leq A(p-\beta)$$

which is true by (2.1) so the proof is complete. □

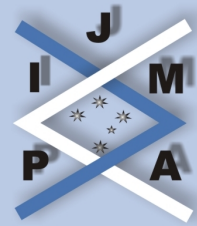
The converse of Theorem 2.1 is also true for functions in $\sigma_p^*(\beta)$, where p is an odd number.

Theorem 2.2. *A function $f(z)$ in σ_p is in $\sigma_p^*(\beta)$ if and only if*

$$(2.2) \quad \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n \leq A(p-\beta).$$

Proof. According to Theorem 2.1 it is sufficient to prove the “only if” part. Suppose that

$$(2.3) \quad \operatorname{Re} \left(\frac{z(\mathcal{J}f(z))'}{(\mathcal{J}f(z))} \right) = \operatorname{Re} \left(\frac{-Apz^{-p} + \sum_{n=p}^{\infty} n(-1)^{n-1} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n}{Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n} \right) < -\beta.$$



By choosing values of z on the real axis so that $\frac{(z(\mathcal{J}f(z))')}{(\mathcal{J}f(z))}$ is real and clearing the denominator in (2.3) and letting $z \rightarrow -1$ through real values we obtain

$$Ap - \sum_{n=p}^{\infty} n \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \geq \beta \left(A + \sum_{n=p}^{\infty} \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \right),$$

which is equivalent to

$$\sum_{n=p}^{\infty} (n + \beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \leq A(p - \beta).$$

□

Corollary 2.3. *If $f(z) \in \sigma_p^*(\beta)$ then*

$$(2.4) \quad a_n \leq \frac{A(p - \beta)(\gamma + n + 1)}{(n + \beta)(\gamma - p + 1)} \quad \text{for } n = p, p + 1, \dots$$

The result is sharp for functions of the type

$$(2.5) \quad f_n(z) = Az^{-p} + (-1)^{n-1} \frac{A(p - \beta)(\gamma + n + 1)}{(n + \beta)(\gamma - p + 1)} z^n.$$

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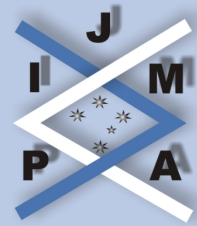
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3. Distortion Bounds and Important Properties of $\sigma_p^*(\beta)$

In this section we obtain distortion bounds for functions in the class $\sigma_p^*(\beta)$ and prove some important properties of this class, where p is an odd number.

Theorem 3.1. Let $f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n$, $a_n \geq 0$ be in the class $\sigma_p^*(\beta)$ and $\beta \geq \gamma + 1$ then

$$(3.1) \quad Ar^{-p} - \frac{A(p-\beta)}{\gamma-p+1} r^p \leq |f(z)| \leq Ar^{-p} + \frac{A(p-\beta)}{\gamma-p+1} r^p.$$

Proof. Since $\beta \geq \gamma + 1$, so $\frac{n+\beta}{\gamma+n+1} \geq 1$. Then

$$(\gamma-p+1) \sum_{n=p}^{\infty} a_n \leq \sum_{n=p}^{\infty} \left(\frac{n+\beta}{\gamma+n+1} \right) (\gamma-p+1) a_n \leq A(p-\beta),$$

and we have

$$\begin{aligned} |f(z)| &= \left| Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n \right| \\ &\leq \frac{A}{r^p} + r^p \sum_{n=p}^{\infty} a_n \leq \frac{A}{r^p} + r^p \frac{A(p-\beta)}{(\gamma-p+1)}. \end{aligned}$$

Similarly,

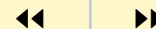
$$|f(z)| \geq \frac{A}{r^p} - \sum_{n=p}^{\infty} a_n r^n \geq \frac{A}{r^p} - r^p \sum_{n=p}^{\infty} a_n \geq \frac{A}{r^p} - \frac{A(p-\beta)}{\gamma-p+1} r^p.$$

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Theorem 3.2. *Let*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad \text{and} \quad g(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} b_n z^n$$

be in the class $\sigma_p^(\beta)$. Then the weighted mean of f and g defined by*

$$W_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)]$$

is also in the same class.

Proof. Since f and g belong to $\sigma_p^*(\beta)$, then by (2.2) we have

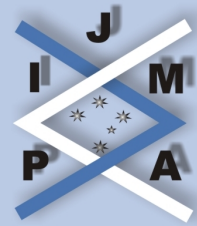
$$(3.2) \quad \begin{cases} \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n \leq A(p-\beta), \\ \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) b_n \leq A(p-\beta). \end{cases}$$

After a simple calculation we obtain

$$W_j(z) = Az^{-p} + \sum_{n=p}^{\infty} \left[\frac{1-j}{2} a_n + \frac{1+j}{2} b_n \right] (-1)^{n-1} z^n.$$

However,

$$\sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) \left[\frac{1-j}{2} a_n + \frac{1+j}{2} b_n \right]$$



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$$\begin{aligned}
 &= \left(\frac{1-j}{2}\right) \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n + \left(\frac{1+j}{2}\right) \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1}\right) b_n \\
 &\stackrel{\text{by (3.2)}}{\leq} \left(\frac{1-j}{2}\right) A(p-\beta) + \left(\frac{1+j}{2}\right) A(p-\beta) \\
 &= A(p-\beta).
 \end{aligned}$$

Hence by Theorem 2.2, $W_j(z) \in \sigma_p^*(\beta)$. □

Theorem 3.3. *Let*

$$f_k(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_{n,k} z^n \in \sigma_p^*(\beta), \quad k = 1, 2, \dots, m$$

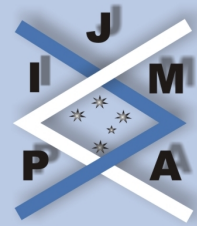
then the arithmetic mean of $f_k(z)$ defined by

$$(3.3) \quad F(z) = \frac{1}{m} \sum_{k=1}^m f_k(z)$$

is also in the same class.

Proof. Since $f_k(z) \in \sigma_p^*(\beta)$, then by (2.2) we have

$$(3.4) \quad \sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_{n,k} \leq A(p-\beta) \quad (k = 1, 2, \dots, m).$$



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After a simple calculation we obtain

$$\begin{aligned} F(z) &= \frac{1}{m} \sum_{k=1}^m \left(Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_{n,k} z^n \right) \\ &= Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left(\frac{1}{m} \sum_{k=1}^m a_{n,k} \right) z^n. \end{aligned}$$

However,

$$\sum_{n=p}^{\infty} (n + \beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) \left(\frac{1}{m} \sum_{k=1}^m a_{n,k} \right) \stackrel{\text{by (3.4)}}{\leq} \frac{1}{m} \sum_{k=1}^m A(p - \beta) = A(p - \beta)$$

which in view of Theorem 2.2 yields the proof of Theorem 3.3. \square

References

- [1] H. IRMAK AND S. OWA, Certain inequalities for multivalent starlike and meromorphically multivalent functions, *Bulletin of the Institute of Mathematics, Academia Sinica*, **31**(1) (2001), 11–21.
- [2] S.B. JOSHI AND H.M. SRIVASTAVA, A certain family of meromorphically multivalent functions, *Computers and Mathematics with Applications*, **38** (1999), 201–211.
- [3] Sh. NAJAFZADEH, A. TEHRANCHI AND S.R. KULKARNI, Application of differential operator on p -valent meromorphic functions, *An. Univ. Oradea Fasc. Mat.*, **12** (2005), 75–90.
- [4] B.A. URALEGADDI AND M. D. GANIGI, Meromorphic starlike functions with alternating coefficients, *Rendiconti di Matematica, Serie VII*, **11** (1991), 441–446.



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