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LOCALIZATION OF FACTORED FOURIER SERIES

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Abstract

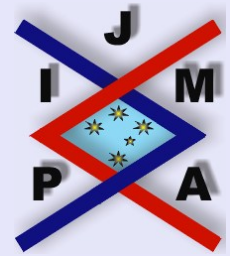
In this paper we deal with a main theorem on the local property of $|\bar{N}, p_n|_k$ summability of factored Fourier series, which generalizes some known results.

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Key words: Absolute summability, Fourier series, Local property.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$(1.1) \quad P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

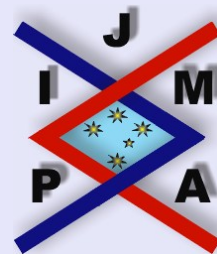
$$(1.2) \quad t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (t_n) of the (\bar{N}, p_n) means of the sequence (s_n) generated by the sequence of coefficients (p_n) .

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, k \geq 1$, if (see [2])

$$(1.3) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1/(n + 1)$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|$ (resp. $|\bar{N}, p_n|$) summability. Also if we take $k = 1$ and $p_n = 1/(n + 1)$, summability $|\bar{N}, p_n|_k$ is equivalent to the summability $|R, \log n, 1|$. A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n , where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.



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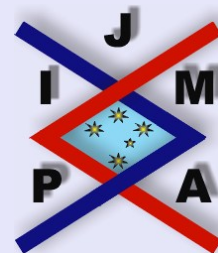
Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$(1.4) \quad \int_{-\pi}^{\pi} f(t) dt = 0$$

and

$$(1.5) \quad f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t).$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f .



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2. Known Results

Mohanty [4] has demonstrated that the $|R, \log n, 1|$ summability of the factored Fourier series

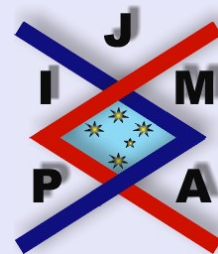
$$(2.1) \quad \sum \frac{A_n(t)}{\log(n+1)}$$

at $t = x$, is a local property of the generating function of f , whereas the $|C, 1|$ summability of this series is not. Matsumoto [3] improved this result by replacing the series (2.1) by

$$(2.2) \quad \sum \frac{A_n(t)}{\{\log \log(n+1)\}^\delta}, \quad \delta > 1.$$

Generalizing the above result Bhatt [1] proved the following theorem.

Theorem A. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*



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3. The Main Result

The aim of the present paper is to prove a more general theorem which includes of the above results as special cases. Also it should be noted that the conditions on the sequence (λ_n) in our theorem, are somewhat more general than in the above theorem.

Now we shall prove the following theorem.

Theorem 3.1. *Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent, then the summability $|\bar{N}, p_n|_k$ of the series $\sum A_n(t) \lambda_n P_n$ at a point is a local property of the generating function f .*

We need the following lemmas for the proof of our theorem.

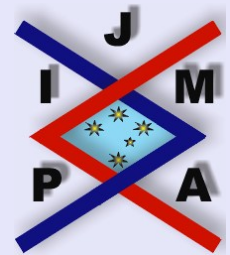
Lemma 3.2. *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \rightarrow \infty$ as $n \rightarrow \infty$, then $P_n \lambda_n = O(1)$ as $n \rightarrow \infty$ and $\sum P_n \Delta \lambda_n < \infty$.*

Proof. Since (λ_n) is non-increasing, we have that

$$P_m \lambda_m = \lambda_m \sum_{n=0}^m p_n = O(1) \sum_{n=0}^m p_n \lambda_n = O(1) \quad \text{as } m \rightarrow \infty.$$

Applying the Abel transform to the sum $\sum_{n=0}^m p_n \lambda_n$, we get that

$$\sum_{n=0}^m P_n \Delta \lambda_n = \sum_{n=0}^m p_n \lambda_n - P_m \lambda_{m+1}.$$



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Since $\lambda_n \geq \lambda_{n+1}$, we obtain

$$\begin{aligned} \sum_{n=0}^m P_n \Delta \lambda_n &\leq P_m \lambda_m + \sum_{n=0}^m p_n \lambda_n \\ &= O(1) + O(1) = O(1) \quad \text{as } m \rightarrow \infty. \end{aligned}$$

□

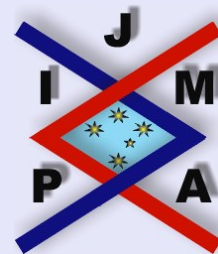
Lemma 3.3. *Let $k \geq 1$ and $s_n = O(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \rightarrow \infty$ as $n \rightarrow \infty$, then the series $\sum a_n \lambda_n P_n$ is summable $|\bar{N}, p_n|_k$.*

Proof. Let (T_n) be the sequence of (\bar{N}, p_n) means of the series $\sum a_n \lambda_n P_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r P_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v \lambda_v P_v.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} P_v a_v \lambda_v.$$



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By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v s_v \Delta \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v p_v \lambda_v \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_{v+1} s_v \lambda_{v+1} + s_n p_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.} \end{aligned}$$

By Minkowski's inequality for $k > 1$, to complete the proof of Lemma 3.3, it is sufficient to show that

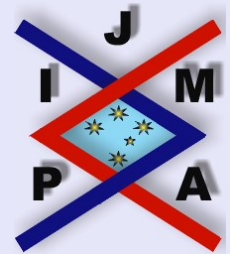
$$(3.1) \quad \sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$

Now, applying Hölder's inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$ and $k > 1$, we get that

$$\begin{aligned} &\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k \\ &\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k P_v P_v \Delta \lambda_v \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \right\}^{k-1}. \end{aligned}$$

Since

$$\sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} P_v \Delta \lambda_v,$$



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it follows by Lemma 3.2 that

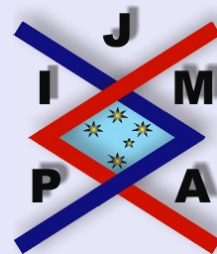
$$\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq \sum_{v=1}^{n-1} P_v \Delta \lambda_v = O(1) \quad \text{as } n \rightarrow \infty.$$

Therefore

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k P_v P_v \Delta \lambda_v \\ &= O(1) \sum_{v=1}^m |s_v|^k P_v P_v \Delta \lambda_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^m P_v \Delta \lambda_v = O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Again

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k (P_v \lambda_v)^k p_v \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k (P_v \lambda_v)^k p_v \\ &= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^k p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \end{aligned}$$



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$$\begin{aligned}
&= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^k \frac{p_v}{P_v} \\
&= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^{k-1} p_v \lambda_v \\
&= O(1) \sum_{v=1}^m p_v \lambda_v = O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

in view of the hypotheses of Theorem 3.1 and Lemma 3.2. Using the fact that $P_v < P_{v+1}$, similarly we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,3}|^k = O(1) \sum_{v=1}^m p_{v+1} \lambda_{v+1} = O(1) \quad \text{as } m \rightarrow \infty.$$

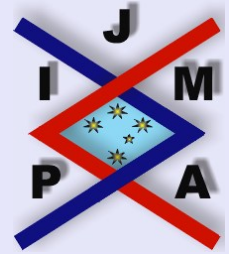
Finally, we have that

$$\begin{aligned}
\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,4}|^k &= \sum_{n=1}^m |s_n|^k (P_n \lambda_n)^{k-1} p_n \lambda_n \\
&= O(1) \sum_{n=1}^m p_n \lambda_n = O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 3.2. Therefore, we get that

$$\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \rightarrow \infty, \quad \text{for } r = 1, 2, 3, 4.$$

This completes the proof of Lemma 3.3. □



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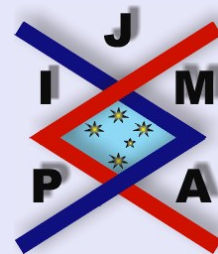
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In the particular case if we take $p_n = 1$ for all values of n in Lemma 3.3, then we get the following corollary.

Corollary 3.4. *Let $k \geq 1$ and and $s_n = O(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum \lambda_n$ is convergent, then the series $\sum na_n\lambda_n$ is summable $|C, 1|_k$.*

Proof of Theorem 3.1. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 3.1 is a consequence of Lemma 3.3. If we take $p_n = 1$ for all values of n in this theorem, then we get a new local property result concerning the $|C, 1|_k$ summability. \square



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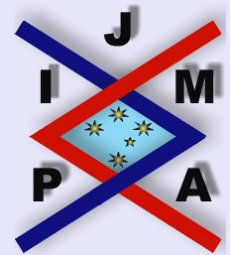
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