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ON INEQUALITIES FOR HYPERGEOMETRIC ANALOGUES OF THE ARITHMETIC-GEOMETRIC MEAN

ROGER W. BARNARD AND KENDALL C. RICHARDS

TEXAS TECH UNIVERSITY
LUBBOCK, TEXAS 79409
roger.w.barnard@ttu.edu

SOUTHWESTERN UNIVERSITY
GEORGETOWN, TEXAS 78626
richards@southwestern.edu

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ABSTRACT. In this note, we present sharp inequalities relating hypergeometric analogues of the arithmetic-geometric mean discussed in [5] and the power mean. The main result generalizes the corresponding sharp inequality for the arithmetic-geometric mean established in [10].

Key words and phrases: Arithmetic-geometric mean, Hypergeometric function, Power mean.

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1. INTRODUCTION

In 1799, Gauss made a remarkable discovery (see equation (1.2) below) regarding the closed form of the compound mean created by iteratively applying the arithmetic mean $\mathcal{A}_1$ and geometric mean $\mathcal{A}_0$, which are special cases of

$$\mathcal{A}_\lambda(a,b) \equiv \left(\frac{a^\lambda + b^\lambda}{2}\right)^{\frac{1}{\lambda}} \quad (\lambda \neq 0),$$

with $\mathcal{A}_0(a,b) \equiv \sqrt{ab}$ for $a, b > 0$. A standard argument reveals that the power mean $\mathcal{A}_\lambda$ is an increasing function of its order λ. In particular, the arithmetic and geometric means satisfy the well-known inequality $\mathcal{A}_0(a,b) \le \mathcal{A}_1(a,b)$. From this it can be shown that the recursively defined sequences given by $a_{n+1} = \mathcal{A}_1(a_n, b_n)$, $b_{n+1} = \mathcal{A}_0(a_n, b_n)$ (with $b_0 = b < a = a_0$) satisfy

$$\mathcal{A}_0(a,b) \le b_n < b_{n+1} < a_{n+1} < a_n \le \mathcal{A}_1(a,b) \quad \text{for all } n \in \mathbb{N}.$$

Thus $\{a_n\}$, $\{b_n\}$ are bounded and monotone sequences satisfying

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \mathcal{A}_1(a_n, b_n) = \lim_{n\to\infty} \mathcal{A}_0(a_n, b_n) = \lim_{n\to\infty} b_{n+1},$$

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by continuity and the fact that these means are strict (i.e. $A_\lambda(a,b) = a$ iff $a = b$). It is this common limit which is used to define the compound mean $\mathcal{A}_1 \otimes \mathcal{A}_0(a,b) \equiv \lim_{n\to\infty} a_n$, commonly referred to as the *arithmetic-geometric mean* $\mathcal{AG} \equiv \mathcal{A}_1 \otimes \mathcal{A}_0$. Moreover, the convergence is quadratic for this particular compound iteration. For more on the historical development of $\mathcal{AG}$, the article [1] by Almkvist and Berndt and the text *Pi and the AGM* by Borwein and Borwein [3] are lively and informative sources.

By construction, $\mathcal{A}_0(a,b) < \mathcal{AG}(a,b) < \mathcal{A}_1(a,b)$ for $a > b > 0$. However, $\mathcal{A}_1$ is not the best possible power mean upper bound for $\mathcal{AG}$. For example, since

$$a_2 = \frac{\frac{a+b}{2} + \sqrt{ab}}{2} = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \mathcal{A}_{1/2}(a,b),$$

it follows that

$$\mathcal{A}_0(a,b) < \mathcal{AG}(a,b) < \mathcal{A}_{1/2}(a,b) \quad \text{for all } a > b > 0.$$

Vamanamurthy and Vuorinen [10] showed that the order $1/2$ is *sharp*. As a result

$$\mathcal{A}_\lambda(a,b) < \mathcal{AG}(a,b) < \mathcal{A}_\mu(a,b) \quad \text{for all } a > b > 0 \tag{1.1}$$

if and only if $\lambda \le 0$ and $\mu \ge 1/2$. The aim of this note is to discuss sharp inequalities that parallel (1.1) for hypergeometric analogues of the arithmetic-geometric mean introduced in [5] and described below.

A review of the above iterative process leading to $\mathcal{AG}$ reveals that any two continuous strict means $\mathcal{M}, \mathcal{N}$ can be used to construct a compound mean, provided $\mathcal{M}$ is comparable to $\mathcal{N}$ (i.e. $\mathcal{M}(a,b) \ge \mathcal{N}(a,b)$ for $a \ge b > 0$). Moreover, $\mathcal{M} \otimes \mathcal{N}$ inherits standard mean properties such as homogeneity (i.e. $\mathcal{M}(sa,sb) = s\mathcal{M}(a,b)$ for $s > 0$) when possessed by both $\mathcal{M}$ and $\mathcal{N}$ (see [3, p. 244]). While the definition of the compound mean as the limit of an iterative process is pleasingly simple, it is natural to pursue a closed-form expression to facilitate further analysis. Gauss engaged in this pursuit for $\mathcal{AG}$ and his discovery yields the following elegant identity (see [3, 9]):

$$\mathcal{AG}(1,r) = \frac{1}{{}_2F_1(1/2, 1/2; 1; 1-r^2)}, \tag{1.2}$$

where ${}_2F_1$ is the Gaussian hypergeometric function

$${}_2F_1(\alpha,\beta;\gamma;z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n, \quad |z| < 1,$$

and $(\alpha)_n \equiv \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \in \mathbb{N}$, $(\alpha)_0 \equiv 1$.

Using modular forms, Borwein et al. (see [5]) constructed quadratically convergent compound means that can be expressed in closed form as

$$\mathcal{M} \otimes \mathcal{N}(1,r) = \frac{1}{{}_2F_1(1/2-s, 1/2+s; 1; 1-r^p)^q}. \tag{1.3}$$

Motivated by a comparison with (1.2), compound means satisfying (1.3) are described in [5] as *hypergeometric analogues* of $\mathcal{AG}$. Sharp inequalities similar to (1.1) for these "close relatives" of $\mathcal{AG}$ can be obtained by applying the following theorem from [8] involving the *hypergeometric mean* ${}_2F_1(-a,b;c;r)^{1/a}$ (discussed by Carlson in [6]) and the weighted power mean given by

$$\mathcal{A}_\lambda(\omega;a,b) \equiv \left[\omega\, a^\lambda + (1-\omega)\, b^\lambda\right]^{1/\lambda} \quad (\lambda \ne 0)$$

and $\mathcal{A}_0(\omega;a,b) \equiv a^\omega b^{1-\omega}$, with weights $\omega, 1-\omega > 0$.

Theorem 1.1 ([8]). *Suppose* $1 \geq a,\ b > 0$ *and* $c > \max\{-a, b\}$. *If* $c \geq \max\{1-2a, 2b\}$, *then*

$$\mathcal{A}_\lambda(1-b/c;1,1-r) \leq {}_2F_1(-a,b;c;r)^{1/a}, \qquad \forall\, r \in (0,1)$$

if and only if $\lambda \leq \frac{a+c}{1+c}$. *If* $c \leq \min\{1-2a, 2b\}$, *then*

$$\mathcal{A}_\lambda(1-b/c;1,1-r) \geq {}_2F_1(-a,b;c;r)^{1/a}, \qquad \forall\, r \in (0,1)$$

if and only if $\lambda \geq \frac{a+c}{1+c}$.

2. MAIN RESULTS

The principal contribution of this note is the observation that Theorem 1.1 can be used to obtain sharp upper bounds for the hypergeometric analogues of $\mathcal{A}G$. We also note that the corresponding lower bounds can be verified directly using elementary series techniques presented here (or as a corollary to more involved developments as in [7]). Simultaneous sharp bounds of this type are of independent interest.

Proposition 2.1. *Suppose* $0 < \alpha \leq 1/2$. *Then for all* $r \in (0,1)$

$$\mathcal{A}_\lambda(\alpha;1,r^\alpha) < \frac{1}{{}_2F_1(\alpha,1-\alpha;1;1-r)} < \mathcal{A}_\mu(\alpha;1,r^\alpha) \tag{2.1}$$

if and only if $\lambda \leq 0$ *and* $\mu \geq (1-\alpha)/(2\alpha)$.

Proof. By the monotonicity of $\lambda \mapsto \mathcal{A}_\lambda$, it suffices to verify the first inequality in (2.1) for the elementary case that $\lambda = 0$. It follows easily by induction that $\frac{(\alpha(1-\alpha))_n}{n!} \geq \frac{(\alpha)_n(1-\alpha)_n}{n!n!}$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}(1-r)^{-\alpha(1-\alpha)} &= \sum_{n=0}^{\infty} \frac{(\alpha(1-\alpha))_n}{n!} r^n \\ &> \sum_{n=0}^{\infty} \frac{(\alpha)_n(1-\alpha)_n}{n!n!} r^n = {}_2F_1(\alpha,1-\alpha;1;r).\end{aligned}$$

This implies

$$\mathcal{A}_0(\alpha;1,(1-r)^\alpha) = (1-r)^{\alpha(1-\alpha)} < {}_2F_1(\alpha,1-\alpha;1;r)^{-1}.$$

The replacement of r by $(1-r)$ completes a proof of the established first inequality in (2.1) for $\lambda \leq 0$. Sharpness follows from the observation that if $\lambda > 0$, then $\mathcal{A}_\lambda(\alpha;1,0) > 0$ while ${}_2F_1(\alpha,1-\alpha;1;r)^{-1} \to 0$ as $r \to 1^-$ (see [9, p. 111]). Thus, for $\lambda > 0$ and r sufficiently close to and less than 1, it follows that

$$\mathcal{A}_\lambda(\alpha;1,(1-r)^\alpha) - {}_2F_1(1/2,1/2;1;r)^{-1} > 0.$$

That is, $\lambda \leq 0$ is necessary and sufficient for the first inequality in (2.1).

The proof of the second inequality is not as obvious. From Theorem 1.1, if $\alpha = -a > 0$, $\beta = 1-\alpha > 0$ and $\max\{\alpha,\beta\} < \gamma \leq \min\{1+2\alpha, 2\beta\}$, then for all $r \in (0,1)$

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;r)^{-1/\alpha} &\leq \left[\left(1-\frac{\beta}{\gamma}\right) + \frac{\beta}{\gamma}(1-r)^\sigma\right]^{\frac{1}{\sigma}} \\ &= \mathcal{A}_\sigma\left(1-\frac{\beta}{\gamma};1,1-r\right)\end{aligned}$$

for the sharp order $\sigma = (\gamma - \alpha)/(1+\gamma)$. (By the proof of Theorem 1.1 in [8], the above inequality is strict unless $\gamma = 1 + 2\alpha = 2\beta$). The conditions for strict inequality are met for $0 < \alpha \le 1/2$, $\beta = 1 - \alpha$, $\gamma = 1$. Thus

$$ {}_2F_1(\alpha, 1-\alpha; 1; 1-r)^{-1} < \mathcal{A}_\sigma(\alpha; 1, r)^\alpha \quad \text{for all } r \in (0,1), $$

if and only if $\sigma \ge (1-\alpha)/2$. Noting that $\mathcal{A}_\sigma(\omega; 1, r)^\alpha = \mathcal{A}_{\sigma/\alpha}(\omega; 1, r^\alpha)$, we obtain the second inequality in (2.1) for $\mu = \sigma/\alpha$. □

Corollary 2.2. *Suppose* $0 < \alpha \le 1/2$ *and* $p > 0$. *Then for all* $r \in (0,1)$

$$ \mathcal{A}_\lambda(\alpha; 1, r) < \frac{1}{{}_2F_1(\alpha, 1-\alpha; 1; 1-r^p)^{\frac{1}{\alpha p}}} < \mathcal{A}_\mu(\alpha; 1, r) \tag{2.2} $$

if and only if $\lambda \le 0$ *and* $\mu \ge p(1-\alpha)/2$.

Proof. Proposition 2.1 implies that for all $r \in (0,1)$ and $q > 0$

$$ \mathcal{A}_{\hat\lambda}(\alpha; 1, r^{p\alpha})^q < \frac{1}{{}_2F_1(\alpha, 1-\alpha; 1; 1-r^p)^q} < \mathcal{A}_{\hat\mu}(\alpha; 1, r^{p\alpha})^q $$

if and only if $\hat\lambda \le 0$ and $\hat\mu \ge (1-\alpha)/(2\alpha)$. Since

$$ \mathcal{A}_{\hat\mu}(\alpha; 1, r^{p\alpha})^q = \mathcal{A}_{\hat\mu/q}(\alpha; 1, r^{pq\alpha}), $$

the result follows by setting $\lambda = \hat\lambda/q$ and $\mu = \hat\mu/q$ for $pq\alpha = 1$. □

It is interesting to note that properties of the important class of *zero-balanced* hypergeometric functions of the form ${}_2F_1(a, b; a+b; \cdot\,)$, which includes those appearing in (2.2), can be applied (see [2, 4]) to obtain inequalities directly relating these compound means.

3. APPLICATIONS

Borwein et al. (see [4, 5] and the references therein) used rather involved modular equations to discover means $\mathcal{M}_n, \mathcal{N}_n$ that can be used to build hypergeometric analogues $\mathcal{AG}_n \equiv \mathcal{M}_n \otimes \mathcal{N}_n$ converging quadratically to closed-form expressions involving ${}_2F_1(1/2 - s, 1/2 + s; 1; \cdot\,)$. In particular, they demonstrated that such compound means exist for $s = 0, 1/6, 1/4, 1/3$ (and the trivial case $s = 1/2$). The resulting closed forms include

$$ \begin{aligned} \mathcal{AG}_2(1, r) &= {}_2F_1(1/2, 1/2; 1; 1-r^2)^{-1}, \\ \mathcal{AG}_3(1, r) &= {}_2F_1(1/3, 2/3; 1; 1-r^3)^{-1}, \\ \mathcal{AG}_4(1, r) &= {}_2F_1(1/4, 3/4; 1; 1-r^2)^{-2}, \\ \mathcal{AG}_6(1, r) &= {}_2F_1(1/6, 5/6; 1; 1-r^3)^{-2}. \end{aligned} $$

Notice that each ${}_2F_1$ satisfies the form appearing in Corollary 2.2. It can be shown that $\mathcal{AG}_2$, $\mathcal{AG}_3$, and $\mathcal{AG}_4$ are formed by compounding the following homogeneous means:

$$ \begin{aligned} \mathcal{M}_2(a, b) &\equiv \frac{a+b}{2}, & \mathcal{N}_2(a, b) &\equiv \sqrt{ab}, \\ \mathcal{M}_3(a, b) &\equiv \frac{a+2b}{3}, & \mathcal{N}_3(a, b) &\equiv \sqrt[3]{\frac{b(a^2+ba+b^2)}{3}}, \\ \mathcal{M}_4(a, b) &\equiv \frac{a+3b}{4}, & \mathcal{N}_4(a, b) &\equiv \sqrt{\frac{b(a+b)}{2}}. \end{aligned} $$

(See [5] for the development of these and the more intricate $\mathcal{M}_6, \mathcal{N}_6$.) Applying Corollary 2.2 with $\alpha = 1/3$, $p = 3$, and invoking homogeneity with $r = b/a$, we find

$$\mathcal{A}_\lambda\left(\frac{1}{3}; a, b\right) < \mathcal{A}G_3(a, b) < \mathcal{A}_\mu\left(\frac{1}{3}; a, b\right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 1$. In a similar fashion, with $\alpha = 1/4$ and $p = 2$, (2.2) implies

$$\mathcal{A}_\lambda\left(\frac{1}{4}; a, b\right) < \mathcal{A}G_4(a, b) < \mathcal{A}_\mu\left(\frac{1}{4}; a, b\right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 3/4$. Since $\mathcal{A}_{3/4}(1/4; a, b) < \mathcal{A}_1(1/4; a, b) = M_4(a, b)$, this sharpens the known fact that $\mathcal{A}G_4(a, b) < M_4(a, b)$. Next, with $\alpha = 1/6$ and $p = 3$, Corollary 2.2 yields

$$\mathcal{A}_\lambda\left(\frac{1}{6}; a, b\right) < \mathcal{A}G_6(a, b) < \mathcal{A}_\mu\left(\frac{1}{6}; a, b\right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 5/4$. Finally, we note that another proof of the sharpness of (1.1) can be obtained by applying Corollary 2.2 with $\alpha = 1/2$ and $p = 2$.

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