

Journal of Inequalities in Pure and Applied Mathematics

HILBERT-PACHPATTE TYPE MULTIDIMENSIONAL INTEGRAL INEQUALITIES

G.D. HANDLEY, J.J. KOLIHA AND J. PEČARIĆ

Department of Mathematics and Statistics
University of Melbourne
Melbourne, VIC 3010
Australia.

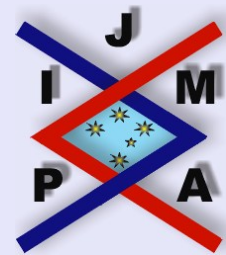
*E*Mail: g.handley@pgrad.unimelb.edu.au

*E*Mail: j.koliha@ms.unimelb.edu.au

Faculty of Textile Engineering
University of Zagreb
10 000 Zagreb
Croatia.

*E*Mail: pecaric@juda.element.hr

©2000 Victoria University
ISSN (electronic): 1443-5756
129-03



volume 5, issue 2, article 34,
2004.

*Received 24 September, 2003;
accepted 01 March, 2004.*

Communicated by: S.S. Dragomir

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Abstract

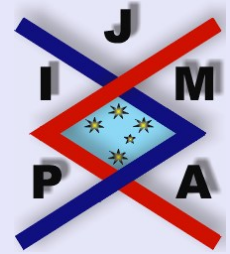
In this paper we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and the present authors.

2000 Mathematics Subject Classification: 26D15

Key words: Hilbert's inequality, Hilbert-Pachpatte integral inequalities, Hölder's inequality.

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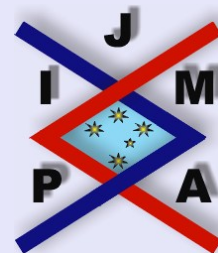
1. Introduction

The integral version of Hilbert's inequality [7, Theorem 316] has been generalized in several directions (see [1, 3, 4, 7, 8, 9, 20, 21, 22]). Recently, inequalities similar to those of Hilbert were considered by Pachpatte in [12, 13, 14, 15, 16, 19]. The present authors in [5, 6] established a new class of related inequalities, which were further extended by Dragomir and Kim [2]. Two and higher dimensional variants were treated by Pachpatte in [17, 18]. In the present paper we use a new systematic approach to these inequalities based on Theorem 3.1, which serves as an abstract springboard to classes of concrete inequalities.

To motivate our investigation, we give a typical result of [17]. In this theorem, $H(I \times J)$ denotes the class of functions $u \in C^{(n-1, m-1)}(I \times J)$ such that $D_1^i u(0, t) = 0$, $0 \leq i \leq n-1$, $t \in J$, $D_2^j u(s, 0) = 0$, $0 \leq j \leq m-1$, $s \in I$, and $D_1^n D_2^{m-1} u(s, t)$ and $D_1^{n-1} D_2^m u(s, t)$ are absolutely continuous on $I \times J$. Here I, J are intervals of the type $I_\xi = [0, \xi)$ for some real $\xi > 0$.

Theorem 1.1 (Pachpatte [17, Theorem 1]). *Let $u(s, t) \in H(I_x \times I_y)$ and $v(k, r) \in H(I_z \times I_w)$. Then, for $0 \leq i \leq n-1$, $0 \leq j \leq m-1$, the following inequality holds:*

$$\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{|D_1^i D_2^j u(s, t) D_1^i D_2^j v(k, r)|}{s^{2n-2i-1} t^{2m-2j-1} + k^{2n-2i-1} r^{2m-2j-1}} dk dr \right) ds dt \\ \leq \frac{1}{2} [A_{i,j} B_{i,j}]^2 \sqrt{xyzw} \left(\int_0^x \int_0^y (x-s)(y-t) |D_1^n D_2^m u(s, t)|^2 ds dt \right)^{\frac{1}{2}} \\ \times \left(\int_0^z \int_0^w (z-k)(w-r) |D_1^n D_2^m v(k, r)|^2 dk dr \right)^{\frac{1}{2}},$$



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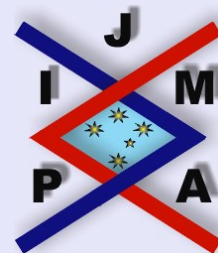
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where

$$A_{ij} = \frac{1}{(n-i-1)!(m-j-1)!}, \quad B_{ij} = \frac{1}{(2n-2i-1)(2m-2j-1)}.$$

The purpose of the present paper is to obtain a simultaneous generalization of Pachpatte's multivariable results [17], and of the results [5, 6] of the present authors. The single variable results [14, 15, 16, 19] follow as special cases of our theorems. Our treatment is based on Theorem 3.1, in particular on the abstract inequality (3.1), which yields a variety of special cases when the functions Φ_i are specified.



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2. Notation and Preliminaries

By \mathbb{Z} (\mathbb{Z}_+) and \mathbb{R} (\mathbb{R}_+) we denote the sets of all (nonnegative) integers and (non-negative) real numbers. We will be working with functions of d variables, where d is a fixed positive integer, writing the variable as a vector $s = (s^1, \dots, s^d) \in \mathbb{R}^d$. A multiindex m is an element $m = (m^1, \dots, m^d)$ of \mathbb{Z}_+^d . As usual, the factorial of a multiindex m is defined by $m! = m^1! \cdots m^d!$. An integer j may be regarded as the multiindex (j, \dots, j) depending on the context. For vectors in \mathbb{R}^d and multiindices we use the usual operations of vector addition and multiplication of vectors by scalars. We write $s \leq \tau$ ($s < \tau$) if $s^j \leq \tau^j$ ($s^j < \tau^j$) for $1 \leq j \leq d$. The same convention will apply to multiindices. In particular, $s \geq 0$ ($s > 0$) will mean $s^j \geq 0$ ($s^j > 0$) for $1 \leq j \leq d$.

If $s = (s^1, \dots, s^d) \in \mathbb{R}^d$ and $s > 0$, we define the *cell*

$$Q(s) = [0, s^1] \times \cdots \times [0, s^j] \times \cdots \times [0, s^d];$$

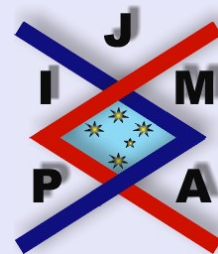
replacing the factor $[0, s^j]$ by $\{0\}$ in this product, we get the *face* $\partial_j Q(s)$ of $Q(s)$.

Let $s = (s^1, \dots, s^d)$, $\tau = (\tau^1, \dots, \tau^d) \in \mathbb{R}^d$, $s, \tau > 0$, let $k = (k^1, \dots, k^d)$ be a multiindex and let $u : Q(s) \rightarrow \mathbb{R}$. Write $D_j = \frac{\partial}{\partial s^j}$. We use the following notation:

$$s^\tau = (s^1)^{\tau^1} \cdots (s^d)^{\tau^d},$$

$$D^k u(s) = D_1^{k^1} \cdots D_d^{k^d} u(s),$$

$$\int_0^s u(\tau) d\tau = \int_0^{s^1} \cdots \int_0^{s^d} u(\tau) d\tau^1 \cdots d\tau^d.$$



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An exponent $\alpha \in \mathbb{R}$ in the expression s^α , where $s \in \mathbb{R}^d$, will be regarded as a multiexponent, that is, $s^\alpha = s^{(\alpha, \dots, \alpha)}$.

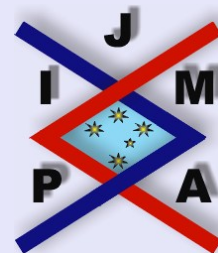
Another positive integer n will be fixed throughout.

The following notation and hypotheses will be used throughout the paper:

$$\begin{aligned}
 I &= \{1, \dots, n\} & n &\in \mathbb{N} \\
 m_i, i \in I & & m_i &= (m_i^1, \dots, m_i^d) \in \mathbb{Z}_+^d \\
 x_i, i \in I & & x_i &= (x_i^1, \dots, x_i^d) \in \mathbb{R}^d, x_i > 0 \\
 p_i, q_i, i \in I & & p_i, q_i &\in \mathbb{R}_+, \frac{1}{p_i} + \frac{1}{q_i} = 1 \\
 p, q & & \frac{1}{p} &= \sum_{i=1}^n \frac{1}{p_i}, \frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i} \\
 a_i, b_i, i \in I & & a_i, b_i &\in \mathbb{R}_+, a_i + b_i = 1 \\
 w_i, i \in I & & w_i &\in \mathbb{R}, w_i > 0, \sum_{i=1}^n w_i = 1.
 \end{aligned}$$

Throughout the paper, u_i, v_i, Φ will denote functions from $[0, x_i]$ to \mathbb{R} of sufficient smoothness. If m is a multiindex and $x \in \mathbb{R}^d, x > 0$, then $C^m[0, x]$ will denote the set of all functions $u : [0, x] \rightarrow \mathbb{R}$ which possess continuous derivatives $D^k u$, where $0 \leq k \leq m$.

The coefficients p_i, q_i are conjugate Hölder exponents used in applications of Hölder's inequality, and the coefficients a_i, b_i are used in exponents to factorize integrands. The coefficients w_i act as weights in applications of the geometric-arithmetic mean inequality; this enables us to pass from products to sums of terms.



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3. The Main Result

First we present a theorem that can be regarded as a template for concrete inequalities obtained by selecting suitable functions Φ_i in (3.1). A special case of this theorem is given in [6, Theorem 3.1].

Theorem 3.1. *Let $v_i, \Phi_i \in C(Q(x_i))$ and let c_i be multiindices for $i \in I$. If*

$$(3.1) \quad |v_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{c_i} \Phi_i(\tau_i) d\tau_i, \quad s_i \in Q(x_i), \quad i \in I,$$

then

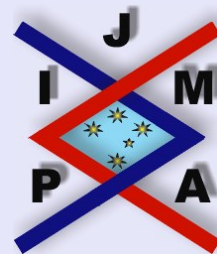
$$(3.2) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

where $\alpha_i = (a_i + b_i q_i) c_i$, $\beta_i = a_i c_i$, and

$$U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i]}.$$

Remark 3.1. *Remembering our conventions, we observe that, for example,*

$$x_i^{1/q_i} = (x_i^1)^{1/q_i} \cdots (x_i^d)^{1/q_i}, \quad \prod_{i=1}^n (\alpha_i + 1)^{1/q_i} = \prod_{i=1}^n \prod_{j=1}^d (\alpha_i^j + 1)^{1/q_i}.$$



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Proof. Factorize the integrand on the right side of (3.1) as

$$(s_i - \tau_i)^{(a_i/q_i+b_i)c_i} \cdot (s_i - \tau_i)^{(a_i/p_i)c_i} \Phi_i(\tau_i)$$

and apply Hölder's inequality [10, p. 106] and Fubini's theorem. Then

$$\begin{aligned} |v_i(s_i)| &\leq \left(\int_0^{s_i} (s_i - \tau_i)^{(a_i+b_iq_i)c_i} d\tau_i \right)^{\frac{1}{q_i}} \\ &\quad \times \left(\int_0^{s_i} (s_i - \tau_i)^{a_i c_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}} \\ &= \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i + 1)^{1/q_i}} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}}. \end{aligned}$$

Using the inequality of means [10, p. 15]

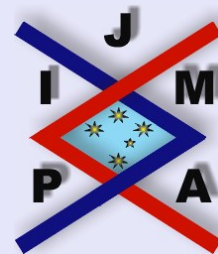
$$\prod_{i=1}^n s_i^{(\alpha_i+1)/q_i} \leq \sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)},$$

we get

$$\prod_{i=1}^n |v_i(s_i)| \leq W \sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)} \prod_{i=1}^n \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}},$$

where

$$W = \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{1/q_i}}.$$



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In the following estimate we apply Hölder's inequality, Fubini's theorem, and, at the end, change the order of integration:

$$\begin{aligned}
 & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
 & \leq W \prod_{i=1}^n \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}} ds_i \right] \\
 & \leq W \prod_{i=1}^n x_i^{1/q_i} \left(\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) ds_i \right)^{\frac{1}{p_i}} \\
 & = \frac{W}{\prod_{i=1}^n (\beta_i + 1)^{1/p_i}} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - \tau_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}}.
 \end{aligned}$$

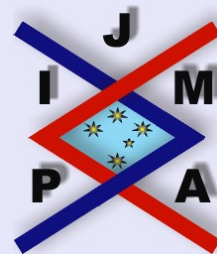
This proves the theorem. \square

If $d = 1$ and v_i are replaced by the derivatives $u_i^{(k)}$, the preceding theorem reduces to [6, Theorem 3.1].

Corollary 3.2. *Under the assumptions of Theorem 3.1,*

$$\begin{aligned}
 (3.3) \quad & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
 & \leq p^{1/p} U \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} ds_i \right)^{\frac{1}{p}},
 \end{aligned}$$

where U is given by (3.2).



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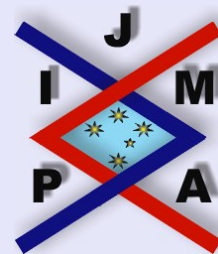
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Proof. By the inequality of means, for any $A_i \geq 0$,

$$\prod_{i=1}^n A_i^{1/p_i} \leq p^{1/p} \left(\sum_{i=1}^n \frac{1}{p_i} A_i \right)^{\frac{1}{p}}.$$

The corollary then follows from the preceding theorem. □

The preceding corollary reduces to [6, Corollary 3.2] in the special case when $d = 1$ and v_i are replaced by $u_i^{(k)}$.



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4. Applications to Derivatives

Convention 1. In this section we shall assume that m_i, k_i are multiindices satisfying $0 \leq k_i \leq m_i - 1$, and write

$$(4.1) \quad \alpha_i = (a_i + b_i q_i)(m_i - k_i - 1), \quad \beta_i = a_i(m_i - k_i - 1).$$

Recall that according to our conventions, $m_i - k_i - 1 = (m_i^1 - k_i^1 - 1, \dots, m_i^d - k_i^d - 1)$.

Theorem 4.1. Let $u_i \in C^{m_i}(Q(x_i))$ be such that $D_j^r u_i(s_i) = 0$ for $s_i \in \partial_j Q(x_i)$, $0 \leq r \leq m_i^j - 1$, $1 \leq j \leq d$, $i \in I$. Then

$$(4.2) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq U_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

where

$$(4.3) \quad U_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)! (\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}.$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11],

$$D^{k_i} u_i(s) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} D^{m_i} u_i(\tau_i) d\tau_i, \quad i \in I.$$



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Inequality (4.2) is proved when we set $v_i(s_i) = D^{k_i}u_i(s_i)$, $c_i = m_i - k_i - 1$, and

$$(4.4) \quad \Phi_i(s_i) = \frac{|D^{m_i}u_i(s_i)|}{(m_i - k_i - 1)!}$$

in Theorem 3.1. □

Corollary 4.2. *Under the hypotheses of Theorem 4.1,*

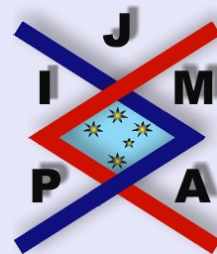
$$(4.5) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq p^{1/p} U_1 \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} |D^{m_i}u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p}},$$

where U_1 is given by (4.3).

Proof. The result follows by applying the inequality of means to the preceding theorem. □

Single variable analogues of the preceding two results were obtained in [6, Theorem 4.1] and [6, Corollary 4.2].

We discuss a number of special cases of Theorem 4.1 with similar examples applying also to Corollary 4.2.



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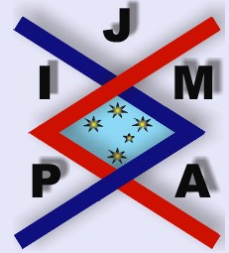


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Example 4.1. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then (4.2) becomes

$$(4.6) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \bar{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

where

$$(4.7) \quad \bar{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(q_i m_i - q_i k_i - q_i + 1)^{1/q_i]}.$$

Example 4.2. If $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$, then

$$(4.8) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^k u_i(s_i)|}{\sum_{i=1}^n s_i^{nm - nk - n + 1}} ds_1 \cdots ds_n \\ \leq \frac{1}{n [(m - k - 1)!]^n (n(m - k - 1) + 1)} \sqrt[n]{x_1 \cdots x_n} \\ \times \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |D^m u_i(s_i)|^{\frac{n}{n-1}} ds_i \right)^{\frac{n-1}{n}}.$$

For $d = 2$ and $q = p = n = 2$ this is Pachpatte's theorem [17, Theorem 1] cited in the Introduction; if $d = 1$ and $q = p = n = 2$, we obtain [14, Theorem 1].

Example 4.3. Let $a_i = 1$ and $b_i = 0$ for $i \in I$. Then (4.2) becomes

$$(4.9) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i - k_i)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \tilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m_i - k_i} |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

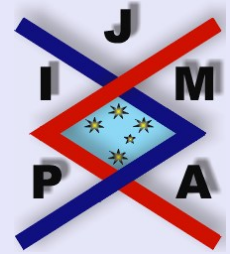
where

$$(4.10) \quad \tilde{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(m_i - k_i)]}.$$

Example 4.4. Set $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (4.2) becomes

$$(4.11) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^k u_i(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n \\ \leq \frac{1}{n [(m - k - 1)!]^n (m - k)^n} \\ \times \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m-k} |D^m u_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.$$

In the following theorem we establish another inequality similar to the integral analogue of Hilbert's inequality.



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Theorem 4.3. Let $u_i \in C^{m_i+1}(Q(x_i))$ be such that $D^{m_i}u_i(s_i) = 0$ for $s_i \in \partial_j Q(s_i)$, $1 \leq j \leq d$, $i \in I$. Then

$$(4.12) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |D^{m_i+1}u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p_i}}.$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11] for $m_i = (0, \dots, 0)$:

$$(4.13) \quad D^{m_i}u_i(s_i) = \int_0^{s_i} D^{m_i+1}u_i(\tau_i) d\tau_i, \quad i \in I.$$

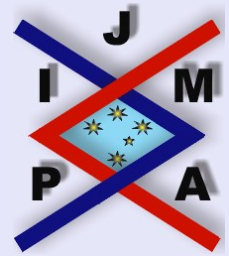
In Theorem 3.1 set $v_i(s_i) = D^{m_i}u_i(s_i)$, $c_i = 0$, $\Phi_i(s_i) = |D^{m_i+1}u_i(s_i)|$, and the result follows. \square

In the special case that $d = 2$, $m_i = (0, 0)$, $p = q = n = 2$, and $w_i = \frac{1}{2}$, the preceding theorem reduces to [17, Theorem 2].

When we apply the inequality of means to the preceding theorem, we get the following corollary which generalizes the inequality obtained in [17, Remark 3].

Corollary 4.4. Under the hypotheses of Theorem 4.3,

$$(4.14) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq p^{1/p} \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i) |D^{m_i+1}u_i(s_i)|^{p_i} ds_i \right)^{\frac{1}{p}}.$$



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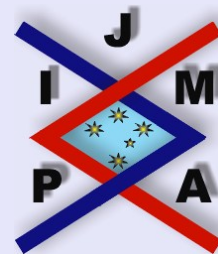
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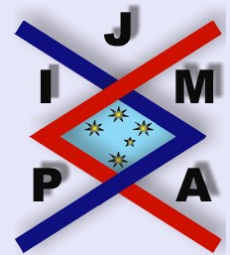
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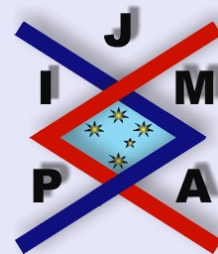
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