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NEW GENERAL INTEGRAL OPERATORS OF p-VALENT FUNCTIONS

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ABSTRACT. In this paper, we introduce new general integral operators. New sufficient conditions for these operators to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order γ $(0 < \gamma \leq 1)$ in the open unit disk are obtained.

Key words and phrases: Analytic functions, p-valent starlike, convex and close-to-convex functions, Uniformly p-valent close-to-convex functions, Strongly starlike, Integral operator.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} \in \{1, 2, \dots\}),$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order β $(0 \leq \beta < p)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{S}_p^\star(\beta)$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}_p$ is said to be p-valently convex of order β $(0 \leq \beta < p)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \qquad (z \in \mathcal{U}).$$

Let $\mathcal{K}_p(\beta)$ denote the class of all those functions which are p-valently convex of order β in $\mathcal{U}$. Furthermore, a function $f(z) \in \mathcal{A}_p$ is said to be in the subclass $\mathcal{C}_p(\beta)$ of p-valently close-to-convex functions of order $\beta(0 \leq \beta < p)$ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \beta \qquad (z \in \mathcal{U}).$$

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Note that $\mathcal{S}_p^\star(0) = \mathcal{S}_p^\star$, $\mathcal{K}_p(0) = \mathcal{K}_p$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the classes of p-valently starlike, p-valently convex and p-valently close-to-convex functions in $\mathcal{U}$. Also, we note that $\mathcal{S}_1^\star = \mathcal{S}^\star$, $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the usual classes of starlike, convex and close-to-convex functions in $\mathcal{U}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}_p(\beta)$ of uniformly p-valent close-to-convex functions of order β $(0 \le \beta < p)$ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)} - \beta\right) \ge \left|\frac{zf'(z)}{g(z)} - p\right| \qquad (z \in \mathcal{U}),$$

for some $g(z) \in \mathcal{US}_p(\beta)$, where $\mathcal{US}_p(\beta)$ is the class of uniformly p-valent starlike functions of order β $(-1 \le \beta < p)$ in $\mathcal{U}$ and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) \ge \left|\frac{zf'(z)}{f(z)} - p\right| \qquad (z \in \mathcal{U}). \tag{1.1}$$

Uniformly p-valent starlike functions were first introduced in [10].

For $\alpha_i > 0$ and $f_i \in \mathcal{A}_p$, we define the following general integral operators

$$F_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt \tag{1.2}$$

and

$$G_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1'(t)}{pt^{p-1}}\right)^{\alpha_1} \cdots \left(\frac{f_n'(t)}{pt^{p-1}}\right)^{\alpha_n} dt. \tag{1.3}$$

If we take $p = 1$, we obtain of the general integral operators $F_1(z) = F_n(z)$ and $G_1(z) = F_{\alpha_1,\dots,\alpha_n}(z)$ introduced and studied by Breaz and Breaz [3] and Breaz et al. [6] (see also [2, 4, 8, 9]). Also for $p = n = 1$, $\alpha_1 = \alpha \in [0,1]$ in (1.2), we obtain the integral operator $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ studied in [12] and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| \le 1/4$ in (1.3), we obtain the integral operator $\int_0^z (f'(t))^\alpha dt$ studied in [11, 15].

There are many papers in which various sufficient conditions for multivalent starlikeness have been obtained. In this paper, we derive new sufficient conditions for the operators $F_p(z)$ and $G_p(z)$ to be p-valently starlike, p-valently close-to-convex and uniformly p-valent close-to-convex in $\mathcal{U}$. Furthermore, we give new sufficient conditions for these two general operators to be strongly starlike of order γ $(0 < \gamma \le 1)$ in $\mathcal{U}$.

In order to derive our main results, we have to recall here the following results:

Lemma 1.1 ([13]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{4} \qquad (z \in \mathcal{U}),$$

then f is p-valently starlike in $\mathcal{U}$.

Lemma 1.2 ([7]). *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} + 1 - p\right| < p + 1 \qquad (z \in \mathcal{U}),$$

then f is p-valently starlike in $\mathcal{U}$.

Lemma 1.3 ([16]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}),$$

where $a > 0$, $b \ge 0$ and $a + 2b \le 1$, then f is p-valently close-to-convex in $\mathcal{U}$.

Lemma 1.4 ([1]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{1}{3} \qquad (z \in \mathcal{U}),$$

then f is uniformly p-valent close-to-convex in $\mathcal{U}$.

Lemma 1.5 ([17]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4}-1 \qquad (z \in \mathcal{U}),$$

then

$$\operatorname{Re}\sqrt{\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

Lemma 1.6 ([14]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > p-\frac{\gamma}{2} \qquad (z \in \mathcal{U}),$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\gamma \qquad (0<\gamma\le 1;\ z \in \mathcal{U}),$$

or f is strongly starlike of order γ in $\mathcal{U}$.

2. SUFFICIENT CONDITIONS FOR THE OPERATOR F_p

We begin by establishing sufficient conditions for the operator F_p to be in $\mathcal{S}_p^{\star}$.

Theorem 2.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1,2,\dots,n$. If $f_i \in \mathcal{A}_p$ for all $i = 1,2,\dots,n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < p+\frac{1}{4\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{2.1}$$

then F_p is p-valently starlike in $\mathcal{U}$.

Proof. From the definition (1.2), we observe that $F_p(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$F_p'(z) = pz^{p-1}\left(\frac{f_1(z)}{z^p}\right)^{\alpha_1}\cdots\left(\frac{f_n(z)}{z^p}\right)^{\alpha_n}. \tag{2.2}$$

Differentiating (2.2) logarithmically and multiplying by z, we obtain

$$\frac{zF_p''(z)}{F_p'(z)} = (p-1)+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-p\right).$$

Thus we have

$$1+\frac{zF_p''(z)}{F_p'(z)} = p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}\right). \tag{2.3}$$

Taking the real part of both sides of (2.3), we have

$$\operatorname{Re}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right) = p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right). \tag{2.4}$$

From (2.4) and (2.1), we obtain

$$\text{Re}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right)<p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(p+\frac{1}{4\sum_{i=1}^{n}\alpha_i}\right)=p+\frac{1}{4}. \tag{2.5}$$

Hence by Lemma 1.1, we get $F_p \in \mathcal{S}_p^\star$. This completes the proof. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we have:

Corollary 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right)<1+\frac{1}{4\alpha} \qquad (z\in\mathcal{U}),$$

where $\alpha > 0,$ then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is starlike in $\mathcal{U}$.

In the next theorem, we derive another sufficient condition for p-valently starlike functions in $\mathcal{U}$.

Theorem 2.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)}-p\right|<\frac{p+1}{\sum_{i=1}^{n}\alpha_i} \qquad (z\in\mathcal{U}), \tag{2.6}$$

then F_p is p-valently starlike in $\mathcal{U}$.

Proof. From (2.3) and the hypotheses (2.6), we have

$$\begin{aligned}\left|1+\frac{zF_p''(z)}{F_p'(z)}-p\right| &= \left|\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-p\right)\right|\\ &< \sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-p\right|\\ &< \sum_{i=1}^{n}\alpha_i\left(\frac{p+1}{\sum_{i=1}^{n}\alpha_i}\right)=p+1.\end{aligned}$$

Now using Lemma 1.2, we immediately get $F_p \in \mathcal{S}_p^\star$. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.3, we have:

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{2}{\alpha} \qquad (z\in\mathcal{U}),$$

where $\alpha > 0,$ then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is starlike in $\mathcal{U}$.

Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for F_p to be p-valently close-to-convex and uniformly p-valent close-to-convex in $\mathcal{U}$.

Theorem 2.5. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\text{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)<p+\frac{(a+b)}{(1+a)(1-b)\sum_{i=1}^{n}\alpha_i} \qquad (z\in\mathcal{U}), \tag{2.7}$$

where $a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then F_p is p-valently close-to-convex in $\mathcal{U}$.

Proof. From (2.4) and the hypotheses (2.7) and applying Lemma 1.3, we have $F_p \in \mathcal{C}_p(\beta)$. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.5, we have:

Corollary 2.6. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{(a+b)}{(1+a)(1-b)\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0, a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ is close-to-convex in $\mathcal{U}$.

Theorem 2.7. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < p + \frac{1}{3\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{2.8}$$

then F_p is uniformly p-valent close-to-convex in $\mathcal{U}$.

Proof. The proof of the theorem follows by applying Lemma 1.4 and using (2.4), (2.8) to get $F_p \in \mathcal{UC}_p(\beta)$. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.7, we have:

Corollary 2.8. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{1}{3\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0,$ then $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ is uniformly close-to-convex in $\mathcal{U}$.

Using Lemma 1.5, we obtain the next result

Theorem 2.9. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > p - \frac{3p+4}{4\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{2.9}$$

then

$$\operatorname{Re}\sqrt{\frac{zF_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

Proof. It follows from (2.4) and (2.9) that

$$\operatorname{Re}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) > p\left(1 - \sum_{i=1}^{n}\alpha_i\right) + \sum_{i=1}^{n}\alpha_i\left(p - \frac{3p+4}{4\sum_{i=1}^{n}\alpha_i}\right) = \frac{p}{4} - 1.$$

By Lemma 1.5, we conclude that

$$\operatorname{Re}\sqrt{\frac{zF_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

□

Letting $n = p = 1,\ \ \alpha_1 = 1$ and $f_1 = f$ in Theorem 2.9, we have:

Corollary 2.10. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > -\frac{3}{4} \qquad (z \in \mathcal{U}), \tag{2.10}$$

then

$$\operatorname{Re}\sqrt{\frac{f(z)}{\int_0^z \left(\frac{f(t)}{t}\right) dt}} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{2.11}$$

3. SUFFICIENT CONDITIONS FOR THE OPERATOR G_p

The first two theorems in this section give a sufficient condition for the integral operator G_p to be in the class $\mathcal{S}_p^\star$.

Theorem 3.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\text{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{1}{4\sum_{i=1}^{n} \alpha_i} \qquad (z \in \mathcal{U}), \tag{3.1}$$

then G_p is p-valently starlike in $\mathcal{U}$.

Proof. From the definition (1.3), we observe that $G_p(z) \in \mathcal{A}_p$ and

$$\frac{zG_p''(z)}{G_p'(z)} = (p-1) + \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)} - (p-1)\right)$$

or

$$1 + \frac{zG_p''(z)}{G_p'(z)} = p\left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.2}$$

Taking the real part of both sides of (3.2), we have

$$\text{Re}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) = p\left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \,\text{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.3}$$

From (3.3) and the hypotheses (3.1), we obtain

$$\text{Re}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) < p\left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \left(p + \frac{1}{4\sum_{i=1}^{n} \alpha_i}\right) = p + \frac{1}{4}. \tag{3.4}$$

Therefore, using Lemma 1.1, it follows that the integral operator G_p belongs to the class $\mathcal{S}_p^\star$. □

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.1, we obtain

Corollary 3.2. *If $f \in \mathcal{A}$ satisfies*

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{1}{4\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is starlike in $\mathcal{U}$.

Theorem 3.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| < \frac{p+1}{\sum_{i=1}^{n} \alpha_i} - p + 1 \qquad (z \in \mathcal{U}), \tag{3.5}$$

where $\sum_{i=1}^{n} \alpha_i > 1$, then G_p is p-valently starlike in $\mathcal{U}$.

Proof. It follows from (3.2) and (3.5) that

$$\begin{aligned} \left|1 + \frac{zG_p''(z)}{G_p'(z)} - p\right| &= \left|\sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right) - (p-1)\sum_{i=1}^{n} \alpha_i\right| \\ &< (p-1)\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \left(\frac{p+1}{\sum_{i=1}^{n} \alpha_i} - p + 1\right) < p + 1. \end{aligned}$$

Therefore, it follows from Lemma 1.2 that G_p is in the class $\mathcal{S}_p^\star$. □

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.3, we obtain:

Corollary 3.4. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{2}{\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0,$ then $\int_0^z (f'(t))^\alpha dt$ is starlike in $\mathcal{U}$.

Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for G_p to be p-valently close-to-convex and uniformly p-valent close-to-convex in $\mathcal{U}$.

Theorem 3.5. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n.$ If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{a+b}{(1+a)(1-b)\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}), \tag{3.6}$$

where $a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then G_p is p-valently close-to-convex in $\mathcal{U}$.

Proof. In view of (3.3) and (3.6) and by using Lemma 1.3, we have $G_p \in \mathcal{C}_p(\beta)$. □

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.5, we obtain

Corollary 3.6. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{a+b}{(1+a)(1-b)\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0,\ a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then $\int_0^z (f'(t))^\alpha dt$ is close-to-convex in $\mathcal{U}$.

Theorem 3.7. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n.$ If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{1}{3\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}), \tag{3.7}$$

then G_p is uniformly p-valent close-to-convex in $\mathcal{U}$.

Proof. In view of (3.3) and (3.7) and by using Lemma 1.4, we have $G_p \in \mathcal{UC}_p(\beta)$. □

Letting $n = p = \alpha = 1$ and $f_1 = f$ in Theorem 3.7, we have:

Corollary 3.8. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{1}{3\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0,$ then $\int_0^z (f'(t))^\alpha dt$ is uniformly close-to-convex in $\mathcal{U}$.

Using Lemma 1.5, we obtain the next result.

Theorem 3.9. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n.$ If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) > p - \frac{3p+4}{4\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}), \tag{3.8}$$

then

$$\operatorname{Re}\sqrt{\frac{zG_p'(z)}{G_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}). \tag{3.9}$$

Proof. It follows from (3.3) and (3.8) that

$$\operatorname{Re}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right) > p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(p-\frac{3p+4}{4\sum_{i=1}^{n}\alpha_i}\right)=\frac{p}{4}-1.$$

By Lemma 1.5, we get the result (3.9). □

Letting $n = p = 1,\ \ \alpha_1 = 1$ and $f_1 = f$ in Theorem 3.9, we have

Corollary 3.10. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) > -\frac{3}{4} \qquad (z \in \mathcal{U}), \tag{3.10}$$

then

$$\operatorname{Re}\sqrt{\frac{zf'(z)}{\int_0^z f'(t)dt}} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{3.11}$$

4. STRONG STARLIKENESS OF THE OPERATORS F_p AND G_p

Applying Lemma 1.6 and using (2.4), we obtain the following sufficient condition for the operator F_p to be strongly starlike of order γ in $\mathcal{U}$.

Theorem 4.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > p-\frac{\gamma}{2\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

then F_p is strongly starlike of order γ $(0 < \gamma \le 1)$ in $\mathcal{U}$.

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 4.1, we have

Corollary 4.2. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 1-\frac{\gamma}{2\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ is strongly starlike of order γ $(0 < \gamma \le 1)$ in $\mathcal{U}$.

Applying once again Lemma 1.6 and using (3.3), we obtain the following sufficient condition for the operator G_p to be strongly starlike of order γ in $\mathcal{U}$.

Theorem 4.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) > p-\frac{\gamma}{2\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

then G_p is strongly starlike of order γ $(0 < \gamma \le 1)$ in $\mathcal{U}$.

Letting $n = p = \alpha_1 = 1$ and $f_1 = f$ in Theorem 4.3, we have

Corollary 4.4. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 1-\frac{\gamma}{2\alpha} \qquad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^{\alpha} dt$ is strongly starlike of order γ $(0 < \gamma \le 1)$ in $\mathcal{U}$.

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