

NOTE ON AN INTEGRAL INEQUALITY APPLICABLE IN PDEs

V. ČULJAK

Department of Mathematics
Faculty of Civil Engineering
University of Zagreb
Kaciceva 26, 10 000 Zagreb, Croatia
EMail: vera@master.grad.hr

Received: 22 April, 2008

Accepted: 06 May, 2008

Communicated by: J. Pečarić

2000 AMS Sub. Class.: Primary 26D15, Secondary 26D99.

Key words: Integral inequality, Nonlinear system of PDEs.

Abstract: The article presents and refines the results which were proven in [1]. We give a condition for obtaining the optimal constant of the integral inequality for the numerical analysis of a nonlinear system of PDEs.



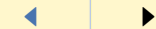
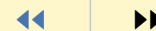
Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

[Title Page](#)

[Contents](#)



Page 1 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

1 Introduction

3

2 Results

4



Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 2 of 12

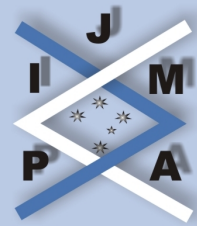
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction

In [1] the following problem is considered and its application to nonlinear system of PDEs is described.

Theorem A. Let $a, b \in \mathbb{R}$, $a < 0$, $b > 0$ and $f \in C[a, b]$, such that: $0 < f \leq 1$ on $[a, b]$, f is decreasing on $[a, 0]$ and

$$\int_a^0 f dx = \int_0^b f dx$$

then

(a) If $p \geq 2$, the inequality

$$(1.1) \quad \int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx$$

holds for all $A_p \geq 2$.

(b) If $1 \leq p < 2$, the inequality

$$(1.2) \quad \int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx$$

holds for all $A_p \geq 4$.

In this note we improve the optimal A_p for the case $1 < p < 2$.

Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 3 of 12

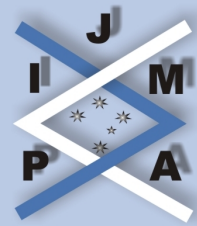
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



2. Results

Theorem 2.1. Let $a, b \in \mathbb{R}$, $a < 0, b > 0$ and $f \in C[a, b]$, such that $0 < f \leq 1$ on $[a, b]$, f is decreasing on $[a, 0]$ and

$$\int_a^0 f dx = \int_0^b f dx.$$

(i) If $a + b \geq 0$, then for $1 \leq p$, this inequality holds

$$(2.1) \quad \int_a^b f^p dx \leq 2 \int_a^{\frac{a+b}{2}} f dx.$$

(ii) If $a + b < 0$ then

(a) If $p \geq 2$, the inequality

$$(2.2) \quad \int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx$$

holds for all $A_p \geq 2$.

(b) If $1 < p < 2$, the inequality

$$(2.3) \quad \int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx$$

holds for all $A_p \geq 2 \frac{1+x_{\max}^{p-1}}{1+x_{\max}}$, where $0 < x_{\max} \leq 1$ is the solution of

$$(2.4) \quad x^{p-1}(p-2) + x^{p-2}(p-1) - 1 = 0.$$

Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

[Title Page](#)

[Contents](#)



Page 4 of 12

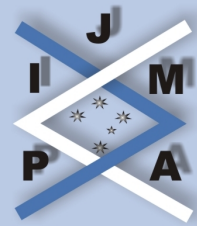
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 5 of 12

Go Back

Full Screen

Close

(c) For $p = 1$ the inequality

$$(2.5) \quad \int_a^b f dx \leq 4 \int_a^{\frac{a+b}{2}} f dx$$

holds.

Proof. As in the proof in [1], we consider two cases: (i) $a + b \geq 0$ and (ii) $a + b < 0$.

(i) First, we suppose that $a + b \geq 0$. Using the properties of the function f , we conclude, for $p \geq 1$, that:

$$\int_a^b f^p dx \leq \int_a^b f dx = 2 \int_a^0 f dx \leq 2 \int_a^{\frac{a+b}{2}} f dx.$$

The constant $A_p = 2$ is the best possible. To prove sharpness, we choose $f = 1$.

(ii) Now we suppose that $a + b < 0$. Let $\varphi : [a, 0] \rightarrow [0, b]$ be a function with the property

$$\int_x^0 f dt = \int_0^{\varphi(x)} f dt.$$

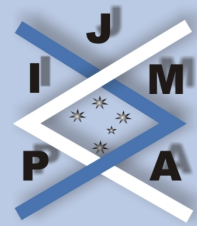
So, $\varphi(x)$ is differentiable and $\varphi(a) = b, \varphi(0) = 0$.

For arbitrary $x \in [a, 0]$, such that $x + \varphi(x) \geq 0$, according to case (i) for $p \geq 1$, we obtain the inequality

$$\int_x^{\varphi(x)} f^p dt \leq 2 \int_x^{\frac{x+\varphi(x)}{2}} f dt.$$

In particular, for $x = a$,

$$\int_a^b f^p dt \leq 2 \int_a^{\frac{a+b}{2}} f dt.$$



Title Page

Contents



Page 6 of 12

Go Back

Full Screen

Close

If we suppose that $x + \varphi(x) < 0$ for arbitrary $x \in [a, 0]$, then we define a new function

$\psi : [a, 0] \rightarrow \mathbb{R}$ by

$$\psi(x) = A_p \int_x^{\frac{x+\varphi(x)}{2}} f dt - \int_x^{\varphi(x)} f^p dt.$$

The function ψ is differentiable and

$$\psi'(x) = \frac{1}{2}A_p(1 + \varphi'(x))f\left(\frac{x + \varphi(x)}{2}\right) - A_p f(x) - f^p(\varphi(x))\varphi'(x) + f^p(x)$$

and $\psi(0) = 0$.

If we prove that $\psi'(x) \leq 0$ then the inequality

$$\int_x^{\varphi(x)} f^p dt \leq A_p \int_x^{\frac{x+\varphi(x)}{2}} f dt$$

holds.

Using the properties of the functions f , φ and the fact that $f(\varphi(x))\varphi'(x) = -f(x)$, we consider $f(\varphi(x))\psi'(x)$ and try to conclude that $f(\varphi(x))\psi'(x) \leq 0$ as follows:

$$\begin{aligned} & f(\varphi(x))\psi'(x) \\ &= f(\varphi(x)) \left[\frac{1}{2}A_p(1 + \varphi'(x))f\left(\frac{x + \varphi(x)}{2}\right) - A_p f(x) - f^p(\varphi(x))\varphi'(x) + f^p(x) \right] \\ &= \frac{1}{2}A_p f(\varphi(x))f\left(\frac{x + \varphi(x)}{2}\right) + \frac{1}{2}A_p f(\varphi(x))\varphi'(x)f\left(\frac{x + \varphi(x)}{2}\right) - A_p f(x)f(\varphi(x)) \\ &\quad - f^p(\varphi(x))\varphi'(x)f(\varphi(x)) + f^p(x)f(\varphi(x)) \end{aligned}$$



Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 7 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-575b

© 2007 Victoria University. All rights reserved.

$$\begin{aligned} &= \frac{1}{2}A_p f(\varphi(x))f\left(\frac{x+\varphi(x)}{2}\right) - \frac{1}{2}A_p f(x)f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x)f(\varphi(x)) \\ &\quad + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \\ &= \frac{1}{2}A_p [f(\varphi(x)) - f(x)]f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x)f(\varphi(x)) \\ &\quad + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)). \end{aligned}$$

For $p \geq 1$, if $[f(\varphi(x)) - f(x)] \leq 0$, then

$$\begin{aligned} &f(\varphi(x))\psi'(x) \\ &\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)]f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x)f(\varphi(x)) \\ &\quad + [f(\varphi(x))f(x) + f(x)f(\varphi(x))] \\ &= \frac{1}{2}A_p [f(\varphi(x)) - f(x)]f\left(\frac{x+\varphi(x)}{2}\right) - (A_p - 2)f(x)f(\varphi(x)). \end{aligned}$$

Then, obviously, $\psi'(x) \leq 0$ for $A_p - 2 \geq 0$.

If we suppose that $[f(\varphi(x)) - f(x)] > 0$ then using the properties of φ , we can conclude that $f\left(\frac{x+\varphi(x)}{2}\right) \leq f(x)$ and we estimate $f(\varphi(x))\psi'(x)$ as follows:

$$\begin{aligned} &f(\varphi(x))\psi'(x) \\ &\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \\ &\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f(\varphi(x))f(x) + f(x)f(\varphi(x)) \end{aligned}$$



Title Page

Contents



Page 8 of 12

Go Back

Full Screen

Close

$$\begin{aligned} &= -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f(\varphi(x))f(x) \leq -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f^2(\varphi(x)) \\ &\leq -\frac{1}{2}(A_p - 4)f^2(\varphi(x)). \end{aligned}$$

So, $\psi'(x) \leq 0$ for $A_p - 4 \geq 0$.

Now, we will consider the sign of $f(\varphi(x))\psi'(x)$ for $p = 1$, $p \geq 2$, and $1 < p < 2$.

(a) For $p \geq 2$, we try to improve the constant $A_p \geq 4$ for the case $a + b < 0$ and

$[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ as follows:

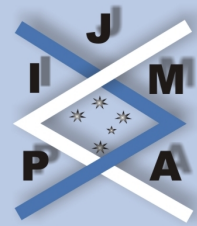
$$\begin{aligned} &f(\varphi(x))\psi'(x) \\ &\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \\ &\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^2(\varphi(x))f(x) + f^2(x)f(\varphi(x)) \\ &\leq \frac{1}{2}f(x)[f(x) + f(\varphi(x))][2f(\varphi(x)) - A_p]. \end{aligned}$$

Hence, $\psi'(x) \leq 0$ for $A_p \geq 2$.

(b) For $1 < p < 2$, we can improve the constant $A_p \geq 4$ for the case $a + b < 0$

and $[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ (for $0 < f(x) = y < f(\varphi(x)) = z \leq 1$), as follows:

$$\begin{aligned} &f(\varphi(x))\psi'(x) \\ &\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \end{aligned}$$



Title Page

Contents



Page 9 of 12

Go Back

Full Screen

Close

$$\begin{aligned} &\leq y \left[-\frac{1}{2}A_p z - \frac{1}{2}A_p y + z^p + y^{p-1}z \right] = y \left[-\frac{1}{2}A_p z \left(1 + \frac{y}{z}\right) + z^p \left(1 + \left(\frac{y}{z}\right)^{p-1}\right) \right] \\ &\leq yz \left[-\frac{1}{2}A_p \left(1 + \frac{y}{z}\right) + 1 + \left(\frac{y}{z}\right)^{p-1} \right]. \end{aligned}$$

So, we conclude that $\psi'(x) \leq 0$ if

$$\left[-\frac{1}{2}A_p(1+t) + 1 + t^{p-1} \right] < 0,$$

for $0 < t = \frac{y}{z} \leq 1$.

Therefore, for $1 < p < 2$ the constant $A_p \geq 2 \max_{0 < t \leq 1} \frac{1+t^{p-1}}{1+t}$.

The function $\frac{1+t^{p-1}}{1+t}$ is concave on $(0, 1]$ and the point t_{\max} where the maximum is achieved is a root of the equation

$$t^{p-1}(p-2) + t^{p-2}(p-1) - 1 = 0.$$

Numerically we get the following values of A_p :

for $p = 1.01$, the constant $A_p \geq 3.8774$,

for $p = 1.99$, the constant $A_p \geq 2.0056$,

for $p = 1.9999$, the constant $A_p \geq 2.0001$.

If we consider the sequence $p_n = 2 - \frac{1}{n}$, then the $\lim_{n \rightarrow \infty} \frac{1+t^{p_n-1}}{1+t} = 1$, but we find that the point t_{\max} where the function $\frac{1+t^{p_n-1}}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = \left(1 - \frac{1+x}{n}\right)^n$.

We use fixed point iteration to find the fixed point for the function $g(x) = \left(1 - \frac{1+x}{n}\right)^n$

$\frac{1+x}{100})^{100}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1})$, $k = 1, 2, \dots, 7$:

$$\begin{aligned}t_0 &= 0.2000000000000000, \\t_1 &= 0.299016021496423, \\t_2 &= 0.270488141422931, \\t_3 &= 0.278419068898826, \\t_4 &= 0.276191402436672, \\t_5 &= 0.276815328895026, \\t_6 &= 0.276640438571483, \\t_7 &= 0.276689450339917.\end{aligned}$$

When $n \rightarrow \infty$, i.e. $p_n \rightarrow 2$, the point t_{\max} where the function $\frac{1+t^{p_n}-1}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = e^{-(1+x)}$.

We use fixed point iteration to find the fixed point for the function $g(x) = e^{-(1+x)}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1})$, $k = 1, 2, \dots, 7$:

$$\begin{aligned}t_0 &= 0.2000000000000000, \\t_1 &= 0.301194211912202, \\t_2 &= 0.272206526577512, \\t_3 &= 0.280212642489384, \\t_4 &= 0.277978184195021, \\t_5 &= 0.278600009316777, \\t_6 &= 0.278426822683543, \\t_7 &= 0.278475046663319\end{aligned}$$



Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 10 of 12

Go Back

Full Screen

Close

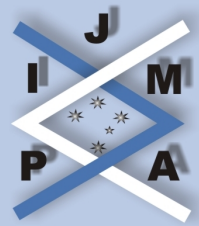
journal of **inequalities**
in pure and applied
mathematics

issn: 1443-575b

If we consider the sequence $p_n = 1 + \frac{1}{n}$ then $\lim_{n \rightarrow \infty} \frac{1+t^{p_n-1}}{1+t} = \frac{2}{1+t}$, and $\sup_{t \in (0,1]} \frac{2}{1+t} = 2$ for $t \rightarrow 0+$.
(c) For $p = 1$,

- if $[f(\varphi(x)) - f(x)] \leq 0$ then $\psi'(x) \leq 0$ for $A_1 - 2 \geq 0$;
- if $[f(\varphi(x)) - f(x)] > 0$ then $\psi'(x) \leq 0$ for $A_1 - 4 \geq 0$,

so, the best constant is $A_1 = 4$. □



Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 11 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] V. JOVANOVIĆ, On an inequality in nonlinear thermoelasticity, *J. Inequal. Pure Appl. Math.*, **8**(4) (2007), Art. 105. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=916>].



Note on an Integral Inequality

V. Čuljak

vol. 9, iss. 2, art. 38, 2008

Title Page

Contents



Page 12 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756