NOTE ON AN INTEGRAL INEQUALITY APPLICABLE IN PDEs

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Abstract: The article presents and refines the results which were proven in [1]. We give a condition for obtaining the optimal constant of the integral inequality for the numerical analysis of a nonlinear system of PDEs.

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1. Introduction

In [1] the following problem is considered and its application to nonlinear system of PDEs is described.

Theorem A. *Let $a, b \in \mathbb{R}$, $a < 0$, $b > 0$ and $f \in C[a,b]$, such that: $0 < f \leq 1$ on $[a,b]$, f is decreasing on $[a,0]$ and*

$$\int_a^0 f\,dx = \int_0^b f\,dx$$

then

(a) If $p \geq 2$, the inequality

$$\int_a^b f^p\,dx \leq A_p \int_a^{\frac{a+b}{2}} f\,dx \tag{1.1}$$

holds for all $A_p \geq 2$.

(b) If $1 \leq p < 2$, the inequality

$$\int_a^b f^p\,dx \leq A_p \int_a^{\frac{a+b}{2}} f\,dx \tag{1.2}$$

holds for all $A_p \geq 4$.

In this note we improve the optimal A_p for the case $1 < p < 2$.

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2. Results

Theorem 2.1. *Let $a, b \in \mathbb{R}$, $a < 0$, $b > 0$ and $f \in C[a, b]$, such that $0 < f \leq 1$ on $[a, b]$, f is decreasing on $[a, 0]$ and*

$$\int_a^0 f dx = \int_0^b f dx.$$

(i) If $a + b \geq 0$, then for $1 \leq p$, this inequality holds

$$\int_a^b f^p dx \leq 2 \int_a^{\frac{a+b}{2}} f dx. \tag{2.1}$$

(ii) If $a + b < 0$ then

(a) If $p \geq 2$, the inequality

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{2.2}$$

holds for all $A_p \geq 2$.

(b) If $1 < p < 2$, the inequality

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{2.3}$$

holds for all $A_p \geq 2 \frac{1+x_{\max}^{p-1}}{1+x_{\max}}$, where $0 < x_{\max} \leq 1$ is the solution of

$$x^{p-1}(p-2) + x^{p-2}(p-1) - 1 = 0. \tag{2.4}$$

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(c) *For* $p = 1$ *the inequality*

$$\int_a^b f\,dx \le 4 \int_a^{\frac{a+b}{2}} f\,dx \tag{2.5}$$

holds.

Proof. As in the proof in [1], we consider two cases: (i) $a+b \ge 0$ and (ii) $a+b < 0$.
(i) First, we suppose that $a+b \ge 0$. Using the properties of the function f, we conclude, for $p \ge 1$, that:

$$\int_a^b f^p dx \le \int_a^b f\,dx = 2\int_a^0 f\,dx \le 2\int_a^{\frac{a+b}{2}} f\,dx.$$

The constant $A_p = 2$ is the best possible. To prove sharpness, we choose $f = 1$.
(ii) Now we suppose that $a+b < 0$. Let $\varphi : [a,0] \to [0,b]$ be a function with the property

$$\int_x^0 f\,dt = \int_0^{\varphi(x)} f\,dt.$$

So, $\varphi(x)$ is differentiable and $\varphi(a) = b, \varphi(0) = 0$.

For arbitrary $x \in [a,0]$, such that $x + \varphi(x) \ge 0$, according to case (i) for $p \ge 1$, we obtain the inequality

$$\int_x^{\varphi(x)} f^p dt \le 2\int_x^{\frac{x+\varphi(x)}{2}} f\,dt.$$

In particular, for $x = a$,

$$\int_a^b f^p dt \le 2\int_a^{\frac{a+b}{2}} f\,dt.$$

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If we suppose that $x+\varphi(x)<0$ for arbitrary $x\in[a,0]$, then we define a new function
$\psi:[a,0]\to\mathbb{R}$ by

$$\psi(x)=A_p\int_x^{\frac{x+\varphi(x)}{2}} f\,dt-\int_x^{\varphi(x)} f^p\,dt.$$

The function ψ is differentiable and

$$\psi'(x)=\frac{1}{2}A_p(1+\varphi'(x))f\left(\frac{x+\varphi(x)}{2}\right)-A_pf(x)-f^p(\varphi(x))\varphi'(x)+f^p(x)$$

and $\psi(0)=0$.

If we prove that $\psi'(x)\le 0$ then the inequality

$$\int_x^{\varphi(x)} f^p\,dt\le A_p\int_x^{\frac{x+\varphi(x)}{2}} f\,dt$$

holds.

Using the properties of the functions f, φ and the fact that $f(\varphi(x))\varphi'(x)=-f(x)$, we consider $f(\varphi(x))\psi'(x)$ and try to conclude that $f(\varphi(x))\psi'(x)\le 0$ as follows:

$$\begin{aligned}
&f(\varphi(x))\psi'(x)\\
&=f(\varphi(x))\left[\frac{1}{2}A_p(1+\varphi'(x))f\left(\frac{x+\varphi(x)}{2}\right)-A_pf(x)-f^p(\varphi(x))\varphi'(x)+f^p(x)\right]\\
&=\frac{1}{2}A_pf(\varphi(x))f\left(\frac{x+\varphi(x)}{2}\right)+\frac{1}{2}A_pf(\varphi(x))\varphi'(x)f\left(\frac{x+\varphi(x)}{2}\right)-A_pf(x)f(\varphi(x))\\
&\qquad-f^p(\varphi(x))\varphi'(x)f(\varphi(x))+f^p(x)f(\varphi(x))
\end{aligned}$$

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$$= \frac{1}{2}A_p f(\varphi(x)) f\left(\frac{x+\varphi(x)}{2}\right) - \frac{1}{2}A_p f(x) f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))$$
$$+ f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x))$$
$$= \frac{1}{2}A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))$$
$$+ f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x)).$$

For $p \geq 1$, if $[f(\varphi(x)) - f(x)] \leq 0$, then

$$f(\varphi(x))\psi'(x)$$
$$\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))$$
$$+ [f(\varphi(x)) f(x) + f(x) f(\varphi(x))]$$
$$= \frac{1}{2}A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - (A_p - 2) f(x) f(\varphi(x)).$$

Then, obviously, $\psi'(x) \leq 0$ for $A_p - 2 \geq 0$.

If we suppose that $[f(\varphi(x)) - f(x)] > 0$ then using the properties of φ, we can conclude that $f\left(\frac{x+\varphi(x)}{2}\right) \leq f(x)$ and we estimate $f(\varphi(x))\psi'(x)$ as follows:

$$f(\varphi(x))\psi'(x)$$
$$\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)] f(x) - A_p f(x) f(\varphi(x)) + f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x))$$
$$\leq \frac{1}{2}A_p [f(\varphi(x)) - f(x)] f(x) - A_p f(x) f(\varphi(x)) + f(\varphi(x)) f(x) + f(x) f(\varphi(x))$$

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$$= -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f(\varphi(x))f(x) \leq -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f^2(\varphi(x))$$
$$\leq -\frac{1}{2}(A_p - 4)f^2(\varphi(x)).$$

So, $\psi'(x) \leq 0$ for $A_p - 4 \geq 0$.

Now, we will consider the sign of $f(\varphi(x))\psi'(x)$ for $p = 1, p \geq 2$, and $1 < p < 2$.

(a) For $p \geq 2$, we try to improve the constant $A_p \geq 4$ for the case $a + b < 0$ and $[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ as follows:

$$f(\varphi(x))\psi'(x)$$
$$\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x))$$
$$\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^2(\varphi(x))f(x) + f^2(x)f(\varphi(x))$$
$$\leq \frac{1}{2}f(x)[f(x) + f(\varphi(x))][2f(\varphi(x)) - A_p].$$

Hence, $\psi'(x) \leq 0$ for $A_p \geq 2$.

(b) For $1 < p < 2$, we can improve the constant $A_p \geq 4$ for the case $a + b < 0$ and $[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ (for $0 < f(x) = y < f(\varphi(x)) = z \leq 1$), as follows:

$$f(\varphi(x))\psi'(x)$$
$$\leq \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x))$$

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$$\leq y\left[-\frac{1}{2}A_p z - \frac{1}{2}A_p y + z^p + y^{p-1}z\right] = y\left[-\frac{1}{2}A_p z\left(1+\frac{y}{z}\right) + z^p\left(1+\left(\frac{y}{z}\right)^{p-1}\right)\right]$$
$$\leq yz\left[-\frac{1}{2}A_p\left(1+\frac{y}{z}\right) + 1 + \left(\frac{y}{z}\right)^{p-1}\right].$$

So, we conclude that $\psi'(x) \leq 0$ if

$$\left[-\frac{1}{2}A_p(1+t) + 1 + t^{p-1})\right] < 0,$$

for $0 < t = \frac{y}{z} \leq 1$.

Therefore, for $1 < p < 2$ the constant $A_p \geq 2\max_{0<t\leq 1}\frac{1+t^{p-1}}{1+t}$.

The function $\frac{1+t^{p-1}}{1+t}$ is concave on $(0,1]$ and the point $t_{\max}$ where the maximum is achieved is a root of the equation

$$t^{p-1}(p-2) + t^{p-2}(p-1) - 1 = 0.$$

Numerically we get the following values of A_p :

$$\begin{array}{ll} \text{for } p = 1.01, & \text{the constant } A_p \geq 3.8774, \\ \text{for } p = 1.99, & \text{the constant } A_p \geq 2.0056, \\ \text{for } p = 1.9999, & \text{the constant } A_p \geq 2.0001. \end{array}$$

If we consider the sequence $p_n = 2 - \frac{1}{n}$, then the $\lim_{n\to\infty}\frac{1+t^{p_n-1}}{1+t} = 1$, but we find that the point $t_{\max}$ where the function $\frac{1+t^{p_n-1}}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = (1 - \frac{1+x}{n})^n$.

We use fixed point iteration to find the fixed point for the function $g(x) = (1 -$

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$\frac{1+x}{100})^{100}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1}), k = 1, 2, ...7$:

$$t_0 = 0.200000000000000,$$
$$t_1 = 0.299016021496423,$$
$$t_2 = 0.270488141422931,$$
$$t_3 = 0.278419068898826,$$
$$t_4 = 0.276191402436672,$$
$$t_5 = 0.276815328895026,$$
$$t_6 = 0.276640438571483,$$
$$t_7 = 0.276689450339917.$$

When $n \to \infty$, i.e. $p_n \to 2$, the point $t_{\max}$ where the function $\frac{1+t^{p_n-1}}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = e^{-(1+x)}$.

We use fixed point iteration to find the fixed point for the function $g(x) = e^{-(1+x)}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1}), k = 1, 2, ...7$:

$$t_0 = 0.200000000000000,$$
$$t_1 = 0.301194211912202,$$
$$t_2 = 0.272206526577512,$$
$$t_3 = 0.280212642489384,$$
$$t_4 = 0.277978184195021,$$
$$t_5 = 0.278600009316777,$$
$$t_6 = 0.278426822683543,$$
$$t_7 = 0.278475046663319$$

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If we consider the sequence $p_n = 1 + \frac{1}{n}$ then $\lim_{n\to\infty} \frac{1+t^{p_n-1}}{1+t} = \frac{2}{1+t}$, and $\sup_{t\in(0,1]} \frac{2}{1+t} = 2$ for $t \to 0+$.

(c) For $p = 1$,

- if $[f(\varphi(x)) - f(x)] \leq 0$ then $\psi'(x) \leq 0$ for $A_1 - 2 \geq 0$;
- if $[f(\varphi(x)) - f(x)] > 0$ then $\psi'(x) \leq 0$ for $A_1 - 4 \geq 0$,

so, the best constant is $A_1 = 4$. □

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