Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 169, 2006

WEIGHTED MULTIPLICATIVE INTEGRAL INEQUALITIES

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Received 12 May, 2006; accepted 20 October, 2006
Communicated by L.-E. Persson

ABSTRACT. We give a generalization of a one-dimensional Carlson type inequality due to G.-S. Yang and J.-C. Fang and a generalization of a multidimensional type inequality due to L. Larsson. We point out the strong and weak parts of each result.

Key words and phrases: Multiplicative integral inequalities, Weights, Carlson's inequality.

2000 *Mathematics Subject Classification.* Primary: 26D15 and Secondary 28A10.

1. INTRODUCTION

Let $(a_n)_{n\geq 1}$ be a non-zero sequence of non-negative numbers and f be a measurable function on $[0,\infty)$. In 1934, F. Carlson [2] proved that the following inequalities

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2, \tag{1.1}$$

$$\left(\int_0^{\infty} f(x)dx\right)^4 \leq \pi^2 \int_0^{\infty} f^2(x)dx \int_0^{\infty} x^2 f^2(x)dx \tag{1.2}$$

hold and $C = \pi^2$ is the best constant in both cases. Several generalizations and applications in different branches of mathematics have been given during the years. For a complete survey

ISSN (electronic): 1443-5756

The research of the second named author was partially supported by a grant of University of Karlstad, Sweden.

We would like to thank to Prof. Lars-Erik Persson and the referee for some generous and useful remarks and comments which have improved the final version of the paper.

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of the results and applications concerning the above inequalities and also interesting historical remarks see the book [5].

G.-S. Yang and J.-C. Fang in [6] proved the following generalization of inequality (1.1)

$$\left(\sum_{n=1}^{\infty} a_n\right)^{2p} < \left(\frac{\pi}{\alpha m}\right)^2 \sum_{n=1}^{\infty} a_n^{p(1+2r-rp)} g^{1-\alpha}(n) \times \sum_{n=1}^{\infty} a_n^{p(1+2r-rp)} g^{1+\alpha}(n) \left(\sum_{n=1}^{\infty} a_n^{rp}\right)^{2(p-2)}, \tag{1.3}$$

when $(a_n)_{n\geq 1}$ is a sequence of nonnegative numbers and g is positive, continuously differentiable, $0 < m = \inf_{x>0} g'(x) < \infty$, $\lim_{x\to\infty} g(x) = \infty$, $p > 2$, $0 < \alpha \leq 1$, $r > 0$.

They also proved in [6] the analogue generalization of the integral inequality (1.2) as follows

$$\left(\int_0^{\infty} f(x)dx\right)^{2p} \leq \left(\frac{\pi}{\alpha m}\right)^2 \int_0^{\infty} f^{p(1+2r-rp)}(x) g^{1-\alpha}(x)dx \times \int_0^{\infty} f^{p(1+2r-rp)}(x) g^{1+\alpha}(x)dx \left(\int_0^{\infty} f^{rp}(x)dx\right)^{2(p-2)}, \tag{1.4}$$

when f is a positive measurable function, g is positive, continuously differentiable and $0 < m = \inf_{x>0} g'(x) < \infty$, $\lim_{x\to\infty} g(x) = \infty$, $p > 2, 0 < \alpha \leq 1, r > 0$.

On the other hand, using another technique, in [3], the following multidimensional extension of the inequality (1.4) was given

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{2p} \leq C\left(\frac{1}{\alpha m^{n/\gamma}}\right)^2 \int_{\mathbb{R}^n} f^{p(1+2r-rp)}(x) g^{(n-\alpha)/\gamma}(x)dx \times \int_{\mathbb{R}^n} f^{q(1+2s-sq)}(x) g^{(n+\alpha)/\gamma}(x)dx \left(\int_{\mathbb{R}^n} f^{rp}(x)dx\right)^{p-2} \left(\int_{\mathbb{R}^n} f^{rq}(x)dx\right)^{q-2}, \tag{1.5}$$

for all positive and measurable functions f. Above, n is a positive integer, r, s are real numbers, $m, \gamma > 0, p, q > 2, 0 < \alpha < n, g : \mathbb{R}^n \to (0,\infty)$ with $g(x) \geq m\,|x|^{\gamma}$, and the constant C does not depend on m, α, γ. This inequality allows a more general setting of parameters and a much larger class of functions g. In [3] an example of admissible function g which is not even continuous was given. It is also shown that the condition $\lim_{x\to\infty} g(x) = \infty$ of (1.4) cannot be relaxed too much, in other words that g cannot be taken essentially bounded. The only weaker point of (1.5) is that it is not given an explicit value of the constant C. We also observe that the proof of (1.4) can be carried on for the value $\alpha = 1$ while this value is not allowed in the proof of (1.5) in the case $n = 1$, which means that Carlson's inequality (1.1) is only a limiting case of (1.5).

In Section 2 of this paper we give two-weight generalizations of the inequalities (1.4) and (1.5). In Section 3 we give a generalization of the discrete inequality (1.3) and some remarks.

2. THE CONTINUOUS CASE

In the next theorem we prove a two-weight generalization of the inequality (1.4).

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a positive measurable function, g_1 and g_2 be positive continuously differentiable and $0 < m = \inf_{x>0}(g_1' g_2 - g_2' g_1) < \infty$. Suppose that $p > 2$ and r is* *an arbitrary real number. Then the following inequality holds*

$$\left(\int_0^\infty f(x)dx\right)^{2p} \le \left(\frac{\pi}{m}\right)^2 \int_0^\infty f^{p(1+2r-rp)}(x)g_1^2(x)dx \times \int_0^\infty f^{p(1+2r-rp)}(x)g_2^2(x)dx \left(\int_0^\infty f^{rp}(x)dx\right)^{2(p-2)}. \tag{2.1}$$

Proof. Observe that the condition $0 < m = \inf_{x>0}(g_1'g_2 - g_2'g_1) < \infty$ implies that $\frac{g_1}{g_2}$ is strictly increasing. Let

$$A = \int_0^\infty f^{p(1+2r-rp)}(x)g_1^2(x)dx \quad \text{and} \quad B = \int_0^\infty f^{p(1+2r-rp)}(x)g_2^2(x)dx,$$

$\lambda > 0$ and q such that $\frac{1}{q} + \frac{1}{p} = 1$. By using Hölder's inequality once for the indices p and q and once for $\frac{p}{q}$ and $\frac{p}{p-q}$ we get

$$\begin{aligned}
\int_0^\infty f(x)dx &\le \left(\int_0^\infty f^q(x)\left(\lambda g_1^2(x) + \frac{1}{\lambda}g_2^2(x)\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \left(\int_0^\infty \frac{1}{\lambda g_1^2(x) + \frac{1}{\lambda}g_2^2(x)}dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{m^{\frac{1}{p}}}\left(\int_0^\infty \frac{\left(\frac{g_1(x)}{g_2(x)}\right)'}{\lambda\left(\frac{g_1(x)}{g_2(x)}\right)^2 + \frac{1}{\lambda}}dx\right)^{\frac{1}{p}} \left(\int_0^\infty f^q(x)\left(\lambda g_1^2(x) + \frac{1}{\lambda}g_2^2(x)\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \\
&= \frac{1}{m^{\frac{1}{p}}}\left(\arctan \frac{g_1(x)}{\lambda g_2(x)}\Big|_0^\infty\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty f^{q-r(p-q)}(x)f^{r(p-q)}\left(\lambda g_1^2(x) + \frac{1}{\lambda}g_2^2(x)\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \\
&\le \left(\frac{\pi}{2m}\right)^{\frac{1}{p}}\left(\int_0^\infty f^{p(1+2r-rp)}(x)\left(\lambda g_1^2(x) + \frac{1}{\lambda}g_2^2(x)\right)dx\right)^{\frac{1}{p}}\left(\int_0^\infty f^{rp}(x)dx\right)^{\frac{p-2}{p}} \\
&= \left(\frac{\pi}{2m}\right)^{\frac{1}{p}}(\lambda A + \frac{1}{\lambda}B)^{\frac{1}{p}}\left(\int_0^\infty f^{rp}(x)dx\right)^{\frac{p-2}{p}}.
\end{aligned}$$

Taking now $\lambda = \sqrt{\frac{B}{A}}$ we get the desired inequality and this completes the proof. □

Remark 2.2. If $rp = 1$ the inequality (2.1) reduces to

$$\left(\int_0^\infty f(x)dx\right)^4 \le \left(\frac{\pi}{m}\right)^2 \int_0^\infty f^2(x)g_1^2(x)dx \int_0^\infty f^2(x)g_2^2(x)dx$$

which becomes (1.2) for $g_1(x) = x$, $g_2(x) = 1$, $x > 0$. The same happens if we let $p \to 2$ in (2.1). If we let $g_1(x) = g^{\frac{1+\alpha}{2}}(x)$, $g_2(x) = g^{\frac{1-\alpha}{2}}(x)$ in (2.1) we get (1.4) which means that (2.1) generalizes also the inequality (4) of [1]. The same inequalities can be given if we replace the interval $[0,\infty)$ by bounded intervals $[a,b]$ or by $(-\infty,\infty)$. On the other hand we can see that it is not necessary to suppose $\inf_{x>0} g_2(x) \ge k > 0$, in other words, the weights $g_2(x) = e^{-x}$ and $g_2(x) = e^x$ are allowed. An interesting case is when $g_2(x) = 1$, $g_1(x) = A_n(x;a) = x(x+na)^{n-1}$, $a > 0$, $n \in \mathbb{N}, n \ge 1$ (Abel polynomials). The inequality

(2.1) becomes

$$\left(\int_0^\infty f(x)dx\right)^4 \le \left(\frac{\pi}{(na)^{n-1}}\right)^2 \int_0^\infty f^2(x)dx \int_0^\infty f^2(x)A_n^2(x;a)dx.$$

To prove a multidimensional extension of the above inequality we need the following lemma which is a special case of Theorem 2 in [4].

Lemma 2.3. *Let $(Z, d\zeta)$ be a measure space on which weights $\beta \ge 0$, $\beta_0 > 0$ and $\beta_1 > 0$ are defined. Suppose that $p_0, p_1 \in (1,2)$ and $\theta \in (0,1)$. Suppose also that there is a constant C such that*

$$\zeta\left(\left\{z : 2^m \le \frac{\beta_0(z)}{\beta_1(z)} < 2^{m+1}\right\}\right) \le C, \qquad m \in \mathbb{Z} \tag{2.2}$$

and that

$$\frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \in L^\infty(\mathbb{Z}, d\zeta).$$

Then there is a constant A such that

$$\|f\beta\|_{L^1(\mathbb{Z},d\zeta)} \le A\,\|f\beta_0\|^\theta_{L^{p_0}(\mathbb{Z},d\zeta)}\,\|f\beta_1\|^{1-\theta}_{L^{p_1}(\mathbb{Z},d\zeta)}. \tag{2.3}$$

The constant A can be chosen of the form $A = A_0 C^{1-\theta/p_0-(1-\theta)/p_1}$, where A_0 does not depend on C.

We are now ready to prove our next multidimensional result which is also a generalization of Theorem 2 of [3]. The technique is similar to that used in the last mentioned theorem. We suppose for simplicity that f is a nonnegative function.

Theorem 2.4. *Let n be a positive integer and $p, q > 2$, $a < 1$ and $r, s \in \mathbb{R}$. Suppose that for some positive constants m, k, the functions $g_1, g_2 : \mathbb{R}^n \to (0,\infty)$ satisfy*

$$g_2(x) \ge m\,|x|^{(nap)/2} \qquad \textit{and} \qquad g_1(x) \ge k\,|x|^{n(p+q-ap)/2}. \tag{2.4}$$

Then there is a constant B independent of m, k, a such that

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{p+q} \le \frac{B}{(1-a)^2m^2k^2}\int_{\mathbb{R}^n} f^{p(1+2r-rp)}(x)g_2^2(x)dx \times \int_{\mathbb{R}^n} f^{q(1+2r-rp)}(x)g_1^2(x)dx \left(\int_{\mathbb{R}^n} f^{rp}(x)dx\right)^{p-2}\left(\int_{\mathbb{R}^n} f^{rq}(x)dx\right)^{q-2}. \tag{2.5}$$

Proof. In Lemma 2.3 put $Z = \mathbb{R}^n$, $d\zeta(x) = \frac{dx}{|x|^n}$, where dx is the Lebesgue measure in $\mathbb{R}^n$, $p_0 = p'$, $p_1 = q'$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Let $\beta(x) = |x|^n$, $\beta_0(x) = |x|^{na}$ and $\beta_1(x) = |x|^{n\frac{1-a\theta}{1-\theta}} = |x|^{n\frac{p+q-ap}{q}}$, where $\theta = \frac{p}{p+q}$.

We observe that

$$\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}} \equiv 1 \in L^\infty(\mathbb{Z}, d\zeta).$$

Also, easy computations give

$$\frac{\beta_0(x)}{\beta_1(x)} = |x|^{\frac{n(a-1)}{1-\theta}} = |x|^{\frac{n(a-1)(p+q)}{q}}.$$

Let

$$\tau = \frac{n(1-a)(p+q)}{q} > 0.$$

Thus $\frac{\beta_0(x)}{\beta_1(x)} \in [2^m, 2^{m+1})$ if and only if $2^{-(m+1)/\tau} \le |x| \le 2^{-m/\tau}$. Using polar coordinates we get

$$\zeta\left(\left\{\frac{\beta_0(x)}{\beta_1(x)} \in [2^m, 2^{m+1})\right\}\right) = \omega_n \int_{2^{-(m+1)/\tau}}^{2^{-m/\tau}} \frac{dr}{r} = \frac{\omega_n \log 2}{\tau},$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. Hence (2.2) holds with $C = \frac{\omega_n \log 2}{\tau}$. Since the conditions of Lemma 2.3 are satisfied, using (2.3) we get

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)dx &= \int_Z f(x)\beta(x)d\zeta(x) \\ &\le A\left(\int_Z (f(x)\beta_0(x))^{p_0}\, d\zeta(x)\right)^{\frac{\theta}{p_0}} \left(\int_Z (f(x)\beta_1(x))^{p_1}\, d\zeta(x)\right)^{\frac{1-\theta}{p_1}} \\ &= A\left(\int_{\mathbb{R}^n} |x|^{nap'} f^{p'}(x)dx\right)^{\frac{p-1}{p+q}} \left(\int_{\mathbb{R}^n} |x|^{n\frac{p+q-ap}{q}q'} f^{q'}(x)dx\right)^{\frac{q-1}{p+q}}.\end{aligned}$$

If we write

$$|x|^{nap'} f^{p'}(x) = \left(|x|^{nap'} f^{p'(1+2r-rp)}(x)\right) f^{p'r(p-2)}(x),$$

$$|x|^{n\frac{p+q-ap}{q}q'} f^{q'}(x) = \left(|x|^{n\frac{p+q-ap}{q}q'} f^{q'(1+2s-sq)}(x)\right) f^{p's(q-2)}(x)$$

and apply Hölder's inequality with $(p-1)$ and $(p-1)/(p-2)$ in the first integral and $(q-1)$ and $(q-1)/(q-2)$ in the second integral we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} f(x)dx\right)^{p+q} &\le A^{p+q}\int_{\mathbb{R}^n} |x|^{nap} f^{p(1+2r-rp)}(x)dx \int_{\mathbb{R}^n} |x|^{n(p+q-ap)} f^{q(1+2s-sq)}(x) \\ &\qquad \times \left(\int_{\mathbb{R}^n} f^{rp}(x)dx\right)^{p-2}\left(\int_{\mathbb{R}^n} f^{sq}(x)dx\right)^{q-2}.\end{aligned}$$

By Lemma 2.3 we can choose $A = A_0\left(\frac{\omega_n \log 2}{\tau}\right)^{2/(p+q)}$, i.e. $A^{p+q} = \frac{B}{(1-a)^2}$, where B does not depend on a. Using (2.4) in estimating the integrals we get the inequality (2.5) and the proof is complete. □

Corollary 2.5. *Let n be a positive integer and $p, q > 2$, $0 < \alpha < n$ and $r, s \in \mathbb{R}$. Suppose that for some positive constants m, γ, the function $g : \mathbb{R}^n \to (0, \infty)$ satisfies*

$$g(x) \ge m\,|x|^{\gamma}. \tag{2.6}$$

Then there is a constant C independent of m, γ, α such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} f(x)dx\right)^{p+q} &\le \frac{C}{\alpha^2 m^{2n/\gamma}}\int_{\mathbb{R}^n} f^{p(1+2r-rp)}(x)g^{(n-\alpha)/\gamma}(x)dx \\ &\quad\times\int_{\mathbb{R}^n} f^{q(1+2r-rp)}(x)g^{(n+\alpha)/\gamma}(x)dx\left(\int_{\mathbb{R}^n} f^{rp}(x)dx\right)^{p-2}\left(\int_{\mathbb{R}^n} f^{rq}(x)dx\right)^{q-2}.\end{aligned}$$

Proof. The condition (2.4) of Theorem 2.4 implies (2.6) if $a = 1 - \frac{\alpha}{np}$, $g_1(x) = g^{(n+\alpha)/2\gamma}(x)$, $g_2(x) = g^{(n-\alpha)/2\gamma}(x)$. □

Remark 2.6. The above corollary is just Theorem 2 of [3]. On the other hand, our Theorem 2.4 is more general than Theorem 2 of [3] since the value $a = 0$ is allowed. This means that g_2 can be taken equivalent with a constant. Thus our inequality can be considered a generalization of Carlson's inequality. In the same way as in [3] one can prove that g_1 cannot be taken essentially bounded. It is also obvious that the condition (2.4) is to some extent weaker than (2.1) although g_2 has to be bounded from below.

3. THE DISCRETE CASE

For completeness we also formulate the discrete case which is a generalization of (1.3).

Theorem 3.1. *Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative numbers and g_1 and g_2 be positive, continuously differentiable functions such that $0 < m = \inf_{x>0}(g_1' g_2 - g_2' g_1) < \infty$, and suppose that g_2 is an increasing function*

$$\left(\sum_{n=1}^{\infty} a_n\right)^{2p} < \left(\frac{\pi}{m}\right)^2 \sum_{n=1}^{\infty} a_n^{p(1+2r-rp)} g_1^2(n) \sum_{n=1}^{\infty} a_n^{p(1+2r-rp)} g_2^2(n) \left(\sum_{n=1}^{\infty} a_n^{rp}\right)^{2(p-2)}. \tag{3.1}$$

Proof. The proof carries on in the same manner as Theorem 2.1. We also use the fact that in the conditions of the hypothesis the function $\frac{1}{\lambda g_1^2(\cdot)+\frac{1}{\lambda}g_2^2(\cdot)}$, $\lambda > 0$ is decreasing and in this case the sum $\sum_{n=1}^{\infty}\left(\lambda g_1^2(n) + \frac{1}{\lambda}g_2^2(n)\right)^{-1}$ can be estimated by the integral $\int_0^{\infty} \frac{1}{\lambda g_1^2(x)+\frac{1}{\lambda}g_2^2(x)}dx$. □

Remark 3.2. Observe the fact that g_2 is an increasing function implies that g_1 is also increasing. If $rp = 1$ then the inequality (3.1) reduces to

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 \leq \left(\frac{\pi}{m}\right)^2 \sum_{n=1}^{\infty} a_n^2 g_1^2(n) \sum_{n=1}^{\infty} a_n^2 g_2^2(n)$$

which becomes (1.1) for $g_1(n) = n$, $g_2(n) = 1$, $n \in \mathbb{N}$. The same is true if we let $p \to 2$ in (3.1). If we let $g_1(x) = g^{\frac{1-\alpha}{2}}(x)$, $g_2(x) = g^{\frac{1+\alpha}{2}}(x)$ in (3.1) we get (1.4) which means that (2.1) generalizes inequality (6) of [6].

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