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A STUDY ON STARLIKE AND CONVEX PROPERTIES FOR HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The objective of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

Key words and phrases: Starlike, Convex, Hypergeometric function, Integral operator.

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1. INTRODUCTION

Let T be the class consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0), \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha, \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda < 1)$ and for all $z \in U$.

Also, let $C(\lambda, \alpha)$ denote the subclass of T consisting of functions which satisfy the condition:

$$\operatorname{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)}\right\} > \alpha, \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda < 1)$ and for all $z \in U$.

From (1.2) and (1.3), we have

$$f(z) \in C(\lambda, \alpha) \Leftrightarrow zf'(z) \in T(\lambda, \alpha). \tag{1.4}$$

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We note that $T(0,\alpha) = T^*(\alpha)$, the class of starlike functions of order α $(0 \le \alpha < 1)$ and $C(0,\alpha) = C(\alpha)$, the class of convex functions of order α $(0 \le \alpha < 1)$ (see Silverman [6]).

Let $F(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \tag{1.5}$$

where $c \neq 0, -1, -2, ...,$ and $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \begin{cases} 1, & n = 0 \\ \theta(\theta+1)\cdots(\theta+n-1) & n \in N = \{1,2,...\}. \end{cases}$$

We note that $F(a,b;c;1)$ converges for $\mathrm{Re}(c-a-b) > 0$ and is related to the Gamma function by

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.6}$$

Silverman [7] gave necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $T^*(\alpha)$ and $C(\alpha)$, also examining a linear operator acting on hypergeometric functions. For other interesting developments on $zF(a,b;c;z)$ in connection with various subclasses of univalent functions, the reader can refer the to works of Carlson and Shaffer [2], Merkes and Scott [4], Ruscheweyh and Singh [5] and Cho et al. [3].

2. MAIN RESULTS

To establish our main results, we need the following lemma due to Altintas and Owa [1].

Lemma 2.1.

(i) *A function $f(z)$ defined by* (1.1) *is in the class $T(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \le 1 - \alpha. \tag{2.1}$$

(ii) *A function $f(z)$ defined by* (1.1) *is in the class $C(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \le 1 - \alpha. \tag{2.2}$$

Theorem A.

(i) *If $a,b > -1, c > 0$ and $ab < 0$, then $zF(a,b;c;z)$ is in $T(\lambda,\alpha)$ if and only if*

$$c > a + b + 1 - \frac{(1-\lambda\alpha)ab}{1-\alpha}. \tag{2.3}$$

(ii) *If $a,b > 0$ and $c > a+b+1$, then $F_1(a,b;c;z) = z[2 - F(a,b;c;z)]$ is in $T(\lambda,\alpha)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(1-\lambda\alpha)ab}{(1-\alpha)(c-a-b-1)}\right] \le 2. \tag{2.4}$$

Proof. (i) Since

$$zF(a,b;c;z) = z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \tag{2.5}$$
$$= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,$$

according to (i) of Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha)\,. \tag{2.6}$$

Note that the left side of (2.6) diverges if $c < a+b+1$. Now

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$
$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{(1-\alpha)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$= (1-\lambda\alpha)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{(1-\alpha)c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

Hence, (2.6) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[(1-\lambda\alpha) + \frac{(1-\alpha)(c-a-b-1)}{ab}\right] \tag{2.7}$$
$$\leq (1-\alpha)\left[\left|\frac{c}{ab}\right| + \frac{c}{ab}\right] = 0.$$

Thus, (2.7) is valid if and only if

$$(1-\lambda\alpha) + \frac{(1-\alpha)(c-a-b-1)}{ab} \leq 0,$$

or equivalently,

$$c \geqslant a+b+1-\frac{(1-\lambda\alpha)ab}{1-\alpha}.$$

(ii) Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\alpha.$$

Now,

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \tag{2.8}$$
$$=\sum_{n=2}^{\infty}[(n-1)(1-\lambda\alpha)+(1-\alpha)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=(1-\lambda\alpha)\sum_{n=1}^{\infty}n\frac{(a)_n(b)_n}{(c)_n(1)_n}+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$=(1-\lambda\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}}+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

Noting that $(\theta)_n=\theta((\theta+1)_{n-1}$ then, (2.8) may be expressed as

$$(1-\lambda\alpha)\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$=(1-\lambda\alpha)\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]$$
$$=(1-\lambda\alpha)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1-\alpha)+\frac{ab(1-\lambda\alpha)}{(c-a-b-1)}\right]-(1-\alpha).$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.4) holds. □

Theorem B.

(i) *If* $a,b>-1$, $ab<0$, *and* $c>a+b+2$, *then* $zF(a,b;c;z)$ *is in* $C(\lambda,\alpha)$ *if and only if*

$$(1-\lambda\alpha)(a)_2(b)_2+(3-2\lambda\alpha-\alpha)ab(c-a-b-2)+(1-\alpha)(c-a-b-2)_2\geqslant 0. \tag{2.9}$$

(ii) *If* $a,b>0$ *and* $c>a+b+2$, *then* $F_1(a,b;c;z)=z[2-F(a,b;c;z)]$ *is in* $C(\lambda,\alpha)$ *if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\frac{(1-\lambda\alpha)(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2}+\left(\frac{3-2\lambda\alpha-\alpha}{1-\alpha}\right)\left(\frac{ab}{c-a-b-1}\right)\right\}\leq 2. \tag{2.10}$$

Proof. (i) Since $zF(a,b;c;z)$ has the form (2.5), we see from (ii) of Lemma 2.1, that our conclusion is equivalent to

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leq\left|\frac{c}{ab}\right|(1-\alpha). \tag{2.11}$$

Note that for $c>a+b+2$, the left side of (2.11) converges. Writing

$$(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)]$$
$$=(n+1)^2(1-\lambda\alpha)+(n+1)(2-\alpha-\lambda\alpha)+(1-\alpha),$$

we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\qquad+ (2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\qquad+ (2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&\quad= (1-\lambda\alpha)\sum_{n=0}^{\infty}n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (3-\alpha-2\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\qquad+ (1-\alpha)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n} \\
&\quad= \frac{(1-\lambda\alpha)(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} \\
&\qquad+ (3-\alpha-2\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&\quad= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[-(1-\lambda\alpha)(a+1)(b+1) \\
&\qquad+(3-\alpha-2\lambda\alpha)(c-a-b-2) + \frac{(1-\alpha)}{ab}(c-a-b-2)_2\Bigg] - \frac{(1-\alpha)c}{ab}.
\end{aligned}$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(1-\alpha)$ if and only if

$$(1-\lambda\alpha)(a+1)(b+1) + (3-\alpha-2\lambda\alpha)(c-a-b-2) + \frac{(1-\alpha)}{ab}(c-a-b-2)_2 \leq 0,$$

which is equivalent to (2.9).

(ii) In view of (ii) of Lemma 2.1, we need to show that

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1-\alpha)\,.$$

Now

$$\begin{aligned}
(2.12)\qquad &\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad= \sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}
\end{aligned}$$

$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - \alpha(1-\lambda)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.$$

Writing $(n+2) = (n+1)+1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} &= \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &\quad= \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + 2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &\quad= \sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + 3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.\end{aligned} \tag{2.14}$$

Substituting (2.13) and (2.14) into the right side of (2.12), yields

$$\begin{aligned}(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}} + (3-2\lambda\alpha-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\ + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.\end{aligned} \tag{2.15}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (2.15) as

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)(a)_2(b)_2}{(c-a-b-2)_2} + \frac{(3-2\lambda\alpha-\alpha)ab}{(c-a-b-1)} + (1-\alpha)\right] - (1-\alpha).$$

By a simplification, we see that the last expression is bounded above by $(1-\alpha)$ if and only if (2.10) holds. □

Putting $\lambda = 0$ in (i) of Theorem B, we have:

Corollary 2.2. *If $a, b > -1$, $ab < 0$, and $c > a+b+2$, then $zF(a,b;c;z)$ is in $C(\alpha)$ if and only if*

$$(a)_2(b)_2 + (3-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geqslant 0.$$

Remark 1. Corollary 2.2, corrects the result obtained by Silverman [7, Theorem 4].

3. AN INTEGRAL OPERATOR

In the theorems below, we obtain similar results in connection with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z) = \int_0^z F(a,b;c;t)dt. \tag{3.1}$$

Theorem C. *Let* $a, b > -1, ab < 0$ *and* $c > \max\{0, a+b\}$. *Then* $G(a,b;c;z)$ *defined by* (3.1) *is in* $T(\lambda,\alpha)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)}{ab}-\frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right]+\frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2}\leq 0. \tag{3.2}$$

Proof. Since

$$G(a,b;c;z)=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n,$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\leq\left|\frac{c}{ab}\right|(1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\\
&\quad=(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}-\alpha(1-\lambda)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}\\
&\quad=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}-\alpha(1-\lambda)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n+1}}\\
&\quad=(1-\lambda\alpha)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}-\alpha(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}\\
&\quad=(1-\lambda\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-\alpha(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}\\
&\quad=(1-\lambda\alpha)\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]\\
&\qquad\qquad-\alpha(1-\lambda)\frac{(c-1)_2}{(a-1)_2(b-1)_2}\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n}\\
&\quad=\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)}{ab}-\frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right]\\
&\qquad\qquad+\frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2}-\frac{(1-\alpha)c}{ab},
\end{aligned}$$

which is bounded above by $(1-\alpha)\left|\frac{c}{ab}\right|$ if and only if (3.2) holds. □

Now, we observe that $G(a,b;c;z)\in C(\lambda,\alpha)$ if and only if $zF(a,b;c;z)\in T(\lambda,\alpha)$. Thus any result of functions belonging to the class $T(\lambda,\alpha)$ about $zF(a,b;c;z)$ leads to that of functions belonging to the class $C(\lambda,\alpha)$. Hence we obtain the following analogous result to Theorem A.

Theorem 3.1. *Let* $a, b > -1, ab < 0$ *and* $c > a+b+2$. *Then* $G(a,b;c;z)$ *defined by* (3.1) *is in* $C(\lambda,\alpha)$ *if and only if*

$$c > a+b+1-\frac{(1-\lambda\alpha)ab}{1-\alpha}.$$

Remark 2. Putting $\lambda = 0$ in the above results, we obtain the results of Silverman [7].

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