

# Journal of Inequalities in Pure and Applied Mathematics

## ON SOME POLYNOMIAL-LIKE INEQUALITIES OF BRENNER AND ALZER

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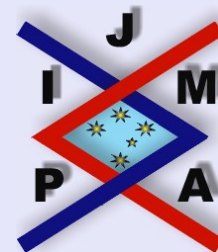
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©2000 Victoria University  
ISSN (electronic): 1443-5756  
135-03



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volume 6, issue 1, article 24,  
2005.

*Received 30 September, 2003;  
accepted 07 November, 2003.*

*Communicated by: T.M. Mills*

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Abstract

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## Abstract

Refinements and extensions are presented for some inequalities of Brenner and Alzer for certain polynomial-like functions.

*2000 Mathematics Subject Classification:* Primary 26D15.

*Key words:* Polynomial inequalities, Switching inequalities, Jensen's inequality

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# 1. Introduction

Brenner [2] has given some interesting inequalities for certain polynomial-like functions. In particular he derived the following.

**Theorem A.** *Suppose  $m > 1$ ,  $0 < p_1, \dots, p_k < 1$  and  $P_k = \sum_{i=1}^k p_i \leq 1$ . Then*

$$(1.1) \quad \sum_{i=1}^k (1 - p_i^m)^m > k - 1 + (1 - P_k)^m.$$

Alzer [1] considered the sum

$$A_k(x, s) = \sum_{i=0}^k \binom{s}{i} x^i (1-x)^{s-i} \quad (0 \leq x \leq 1)$$

and proved the following companion inequality to (1.1).

**Theorem B.** *Let  $p, q, m$  and  $n$  be positive real numbers and  $k$  a nonnegative integer. If  $p + q \leq 1$  and  $m, n > k + 1$ , then*

$$(1.2) \quad A_k(p^m, n) + A_k(q^n, m) > 1 + A_k((p+q)^{\min(m,n)}, \max(m,n)).$$

In the special case  $k = 0$  this provides

$$(1.3) \quad (1 - p^m)^n + (1 - q^n)^m > 1 + (1 - (p+q)^{\min(m,n)})^{\max(m,n)} \quad \text{for } p, q > 0.$$

In Section 2 we use (1.3) to derive an improvement of Theorem A and a corresponding version of Theorem B. In Section 3 we give a related Jensen inequality and concavity result.



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## 2. Basic Results

**Theorem 2.1.** *Under the conditions of Theorem A we have*

$$(2.1) \quad \sum_{i=1}^k (1 - p_i^m)^m > k - 1 + (1 - P_k^m)^m.$$

*Proof.* We proceed by mathematical induction, (1.3) with  $n = m$  providing a basis

$$(2.2) \quad (1 - p^m)^m + (1 - q^m)^m > 1 + (1 - (p+q)^m)^m \quad \text{for } p, q > 0 \text{ and } p + q \leq 1$$

for  $k = 2$ . For the inductive step, suppose that (2.1) holds for some  $k \geq 2$ , so that

$$\begin{aligned} \sum_{i=1}^{k+1} (1 - p_i^m)^m &= \sum_{i=1}^k (1 - p_i^m)^m + (1 - p_{k+1}^m)^m \\ &> k - 1 + (1 - P_k^m)^m + (1 - p_{k+1}^m)^m. \end{aligned}$$

Applying (2.2) yields

$$\begin{aligned} \sum_{i=1}^{k+1} (1 - p_i^m)^m &> k - 1 + 1 + (1 - (P_k + p_{k+1})^m)^m \\ &= k + (1 - P_{k+1}^m)^m. \end{aligned}$$

□



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For the remaining results in this paper it is convenient, for a fixed nonnegative integer  $k$  and  $m > k + 1$ , to define

$$B(x) := A_k(x^m, m).$$

**Theorem 2.2.** *Let  $p_1, \dots, p_\ell$  and  $m$  be positive real numbers. If*

$$P_\ell := \sum_{i=1}^{\ell} p_i,$$

then

$$(2.3) \quad \sum_{j=1}^{\ell} B(p_j) > \ell - 1 + B(P_\ell).$$

*Proof.* We establish the result by induction, (1.2) with  $n = m$  providing a basis

$$(2.4) \quad B(p) + B(q) > 1 + B(p + q) \quad \text{for } p, q > 0 \text{ and } p + q \leq 1$$

for  $\ell = 2$ . Suppose (2.3) to be true for some  $\ell \geq 2$ . Then by the inductive hypothesis

$$\begin{aligned} \sum_{j=1}^{\ell+1} B(p_j) &= \sum_{j=1}^{\ell} B(p_j) + B(p_{\ell+1}) \\ &> \ell - 1 + B(P_\ell) + B(p_{\ell+1}). \end{aligned}$$



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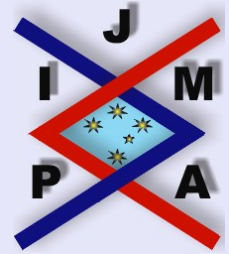
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Now applying (2.4) yields

$$(2.5) \quad \sum_{j=1}^{\ell+1} B(p_j) > \ell - 1 + 1 + B(P_\ell + p_{\ell+1}) \\ = \ell + B(P_{\ell+1})$$

as desired. □



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### 3. Concavity of $B$

Inequality (2.3) is of the form

$$\sum_{j=1}^n f(p_j) > (n-1)f(0) + f\left(\sum_{j=1}^n p_i\right),$$

that is, the Petrović inequality for a concave function  $f$ . A natural question is whether  $B$  satisfies the corresponding Jensen inequality

$$(3.1) \quad B\left(\frac{1}{n}\sum_{j=1}^n p_j\right) \geq \frac{1}{n}\sum_{j=1}^n B(p_j)$$

for positive  $p_1, p_2, \dots, p_n$  satisfying  $\sum_{j=1}^n p_j \leq 1$  and indeed whether  $B$  is concave. We now address these questions. It is convenient to first deal separately with the case  $n = 2$ .

**Theorem 3.1.** *Suppose  $p, q$  are positive and distinct with  $p + q \leq 1$ . Then*

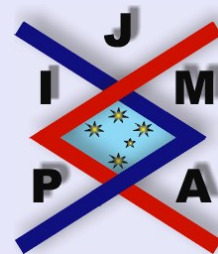
$$(3.2) \quad B\left(\frac{p+q}{2}\right) > \frac{1}{2}[B(p) + B(q)].$$

*Proof.* Let  $u \in [0, 1)$ . For  $p \in [0, 1 - u]$  we define

$$G(p) = B(p) + B(1 - u - p).$$

By an argument of Alzer [1] we have

$$(3.3) \quad G'(p) = \binom{m}{k} (m-k)mp^{m-1}(1-p^m)^{m-1} \left(\frac{p^m}{1-p^m}\right)^k [g(p) - 1],$$



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where

$$(3.4) \quad g(p) = \left( \frac{1-u-p}{1-p^m} \right)^{m-1} \left( \frac{1-(1-u-p)^m}{p} \right)^{m-1} \\ \times \left( \frac{(1-u-p)^m}{1-(1-u-p)^m} \right)^k \left( \frac{1-p^m}{p^m} \right)^k$$

is a strictly decreasing function.

It was shown in [1] that there exists  $p_0 \in (0, 1-u)$  such that  $G(p)$  is strictly increasing on  $[0, p_0]$  and strictly decreasing on  $[p_0, 1-u]$ , so that

$$G(p) < G(p_0) \quad \text{for } p \in [0, 1-u], p \neq p_0.$$

On the other hand, we have by (3.4) that  $g((1-u)/2) = 1$  and so from (3.3)  $G'((1-u)/2) = 0$ . Hence  $p_0 = (1-u)/2$  and therefore

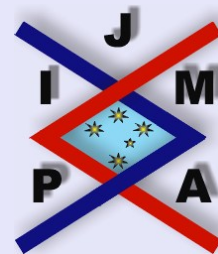
$$G(p) < G\left(\frac{1-u}{2}\right) \quad \text{for } p \neq (1-u)/2.$$

Set  $u = 1 - (p + q)$ . Since  $p \neq q$ , we must have  $p \neq (1-u)/2$ . Therefore

$$G(p) < G\left(\frac{p+q}{2}\right),$$

which is simply (3.2). □

**Corollary 3.2.** *The map  $B$  is concave on  $(0, 1)$ .*



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*Proof.* Theorem 3.1 gives that  $B$  is Jensen concave, so that  $-B$  is Jensen-convex. Since  $B$  is continuous, we have by a classical result [3, Chapter 3] that  $-B$  must also be convex and so  $B$  is concave.  $\square$

The following result finishes additional information about strictness.

**Theorem 3.3.** *Let  $p_1, \dots, p_n$ , be positive numbers with  $\sum_{j=1}^n p_j \leq 1$ . Then (3.1) applies. If not all the  $p_j$  are equal, then the inequality is strict.*

*Proof.* The result is trivial with equality if the  $p_j$  all share a common value, so we assume at least two different values.

We proceed by induction, Theorem 3.1 providing a basis for  $n = 2$ . For the inductive step, suppose that (3.1) holds for some  $n \geq 2$  and that  $\sum_{j=1}^{n+1} p_j \leq 1$ . Without loss of generality we may assume that  $p_{n+1}$  is the greatest of the values  $p_j$ . Since not all the values  $p_j$  are equal, we therefore have

$$p_{n+1} > \frac{1}{n} \sum_{j=1}^n p_j.$$

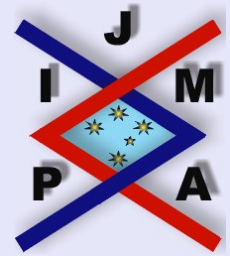
This rearranges to give

$$\frac{1}{n} \sum_{j=1}^n p_j < \frac{1}{n} \left[ p_{n+1} + \frac{n-1}{n+1} \sum_{j=1}^{n+1} p_j \right].$$

Both sides of this inequality take values in  $(0, 1)$ .

Also we have

$$\frac{1}{n+1} \sum_{j=1}^{n+1} p_j = \frac{1}{2} \left[ \frac{1}{n} \sum_{j=1}^n p_j + \frac{1}{n} \left\{ p_{n+1} + \frac{n-1}{n+1} \sum_{j=1}^{n+1} p_j \right\} \right].$$



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Hence applying (3.2) provides

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{2}\left[B\left(\frac{1}{n}\sum_{j=1}^np_j\right) + B\left(\frac{1}{n}\left\{p_{n+1} + \frac{n-1}{n+1}\sum_{j=1}^{n+1}p_j\right\}\right)\right].$$

By the inductive hypothesis

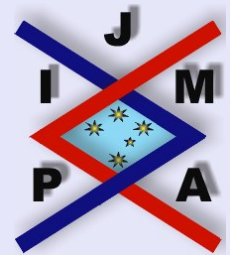
$$B\left(\frac{1}{n}\sum_{j=1}^np_j\right) \geq \frac{1}{n}\sum_{j=1}^nB(p_j)$$

and

$$B\left(\frac{1}{n}\left\{p_{n+1} + \frac{n-1}{n+1}\sum_{j=1}^{n+1}p_j\right\}\right) \geq \frac{1}{n}\left[B(p_{n+1}) + (n-1)B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right)\right].$$

Hence

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{2n}\left[\sum_{j=1}^{n+1}B(p_j) + (n-1)B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right)\right].$$



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Rearrangement of this inequality yields

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{n+1}\sum_{j=1}^{n+1}B(p_j),$$

the desired result. □

**Remark 1.** Taken together, relations (2.5) and (3.1) give

$$(3.5) \quad n-1 + B\left(\sum_{j=1}^n p_j\right) < \sum_{j=1}^n B(p_j) \leq nB\left(\frac{1}{n}\sum_{j=1}^n p_j\right),$$

the second inequality being strict unless all the values  $p_j$  are equal. If  $\sum_{j=1}^n p_j = 1$ , this simplifies to

$$(3.6) \quad n-1 < \sum_{j=1}^n B(p_j) \leq nB(n^{-1}),$$

since  $B(1) = 0$ .

For  $k = 0$ , (3.5) and (3.6) become (for  $m > 1$ ) respectively

$$n-1 + \left(1 - \left(\sum_{j=1}^n p_j\right)^m\right)^m < \sum_{j=1}^n (1 - p_j^m)^m \leq n \left(1 - \left(\frac{1}{n}\sum_{j=1}^n p_j\right)^m\right)^m$$

and

$$n-1 < \sum_{j=1}^n (1 - p_j^m)^m \leq n(1 - n^{-m})^m.$$



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