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ON SOME POLYNOMIAL–LIKE INEQUALITIES OF BRENNER AND ALZER

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Abstract

Refinements and extensions are presented for some inequalities of Brenner and Alzer for certain polynomial–like functions.

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Contents

1. Introduction

Brenner [2] has given some interesting inequalities for certain polynomial–like functions. In particular he derived the following.

Theorem A. *Suppose* $m > 1$, $0 < p_1, \ldots, p_k < 1$ *and* $P_k = \sum_{i=1}^{k} p_i \leq 1$. *Then*

$$\sum_{i=1}^{k} (1 - p_i^m)^m > k - 1 + (1 - P_k)^m . \tag{1.1}$$

Alzer [1] considered the sum

$$A_k(x, s) = \sum_{i=0}^{k} \binom{s}{i} x^i (1 - x)^{s-i} \quad (0 \leq x \leq 1)$$

and proved the following companion inequality to (1.1).

Theorem B. *Let* p, q, m *and* n *be positive real numbers and* k *a nonnegative integer. If* $p + q \leq 1$ *and* $m, n > k + 1$, *then*

$$A_k(p^m, n) + A_k(q^n, m) > 1 + A_k((p + q)^{\min(m,n)}, \max(m, n)). \tag{1.2}$$

In the special case $k = 0$ this provides

$$(1-p^m)^n + (1-q^n)^m > 1 + (1-(p+q)^{\min(m,n)})^{\max(m,n)} \quad \text{for } p, q > 0. \tag{1.3}$$

In Section 2 we use (1.3) to derive an improvement of Theorem A and a corresponding version of Theorem B. In Section 3 we give a related Jensen inequality and concavity result.

2. Basic Results

Theorem 2.1. *Under the conditions of Theorem A we have*

$$\sum_{i=1}^{k}(1-p_i^m)^m > k-1+(1-P_k^m)^m. \tag{2.1}$$

Proof. We proceed by mathematical induction, (1.3) with $n = m$ providing a basis

$$(1-p^m)^m+(1-q^m)^m > 1+(1-(p+q)^m)^m \quad \text{for } p,q>0 \text{ and } p+q\le 1 \tag{2.2}$$

for $k = 2$. For the inductive step, suppose that (2.1) holds for some $k \geq 2$, so that

$$\begin{aligned}\sum_{i=1}^{k+1}(1-p_i^m)^m &= \sum_{i=1}^{k}(1-p_i^m)^m + (1-p_{k+1}^m)^m \\ &> k-1+(1-P_k^m)^m+(1-p_{k+1}^m)^m.\end{aligned}$$

Applying (2.2) yields

$$\begin{aligned}\sum_{i=1}^{k+1}(1-p_i^m)^m &> k-1+1+(1-(P_k+p_{k+1})^m)^m \\ &= k+\left(1-P_{k+1}^m\right)^m.\end{aligned}$$

□

For the remaining results in this paper it is convenient, for a fixed nonnegative integer k and $m > k+1$, to define

$$B(x) := A_k\left(x^m, m\right).$$

Theorem 2.2. *Let $p_1, \dots, p_\ell$ and m be positive real numbers. If*

$$P_\ell := \sum_{i=1}^{\ell} p_i,$$

then

$$\sum_{j=1}^{\ell} B(p_j) > \ell - 1 + B\left(P_\ell\right). \tag{2.3}$$

Proof. We establish the result by induction, (1.2) with $n = m$ providing a basis

$$B(p) + B(q) > 1 + B(p+q) \quad \text{for } p, q > 0 \text{ and } p + q \leq 1 \tag{2.4}$$

for $\ell = 2$. Suppose (2.3) to be true for some $\ell \geq 2$. Then by the inductive hypothesis

$$\sum_{j=1}^{\ell+1} B(p_j) = \sum_{j=1}^{\ell} B(p_j) + B(p_{\ell+1})$$
$$> \ell - 1 + B(P_\ell) + B(p_{\ell+1}).$$

Now applying (2.4) yields

$$\sum_{j=1}^{\ell+1} B(p_j) > \ell - 1 + 1 + B(P_\ell + p_{\ell+1}) = \ell + B(P_{\ell+1}) \tag{2.5}$$

as desired. □

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3. Concavity of B

Inequality (2.3) is of the form

$$\sum_{j=1}^{n} f(p_j) > (n-1)f(0) + f\left(\sum_{j=1}^{n} p_i\right),$$

that is, the Petrović inequality for a concave function f. A natural question is whether B satisfies the corresponding Jensen inequality

$$B\left(\frac{1}{n}\sum_{j=1}^{n} p_j\right) \geq \frac{1}{n}\sum_{j=1}^{n} B(p_j) \tag{3.1}$$

for positive $p_1, p_2, \ldots, p_n$ satisfying $\sum_{j=1}^{n} p_j \leq 1$ and indeed whether B is concave. We now address these questions. It is convenient to first deal separately with the case $n = 2$.

Theorem 3.1. *Suppose p, q are positive and distinct with $p + q \leq 1$. Then*

$$B\left(\frac{p+q}{2}\right) > \frac{1}{2}\left[B(p) + B(q)\right]. \tag{3.2}$$

Proof. Let $u \in [0, 1)$. For $p \in [0, 1-u]$ we define

$$G(p) = B(p) + B(1-u-p).$$

By an argument of Alzer [1] we have

$$G'(p) = \binom{m}{k}(m-k)mp^{m-1}(1-p^m)^{m-1}\left(\frac{p^m}{1-p^m}\right)^k [g(p)-1], \tag{3.3}$$

where

$$g(p)=\left(\frac{1-u-p}{1-p^m}\right)^{m-1}\left(\frac{1-(1-u-p)^m}{p}\right)^{m-1} \times\left(\frac{(1-u-p)^m}{1-(1-u-p)^m}\right)^k\left(\frac{1-p^m}{p^m}\right)^k \tag{3.4}$$

is a strictly decreasing function.

It was shown in [1] that there exists $p_0 \in (0, 1-u)$ such that $G(p)$ is strictly increasing on $[0, p_0]$ and strictly decreasing on $[p_0, 1-u]$, so that

$$G(p) < G(p_0) \quad \text{for} \quad p \in [0, 1-u],\ p \neq p_0.$$

On the other hand, we have by (3.4) that $g((1-u)/2) = 1$ and so from (3.3) $G'((1-u)/2) = 0$. Hence $p_0 = (1-u)/2$ and therefore

$$G(p) < G\left(\frac{1-u}{2}\right) \quad \text{for} \quad p \neq (1-u)/2.$$

Set $u = 1-(p+q)$. Since $p \neq q$, we must have $p \neq (1-u)/2$. Therefore

$$G(p) < G\left(\frac{p+q}{2}\right),$$

which is simply (3.2). □

Corollary 3.2. *The map B is concave on $(0, 1)$.*

Proof. Theorem 3.1 gives that B is Jensen concave, so that $-B$ is Jensen–convex. Since B is continuous, we have by a classical result [3, Chapter 3] that $-B$ must also be convex and so B is concave. □

The following result funishes additional information about strictness.

Theorem 3.3. *Let $p_1, \ldots, p_n$, be positive numbers with $\sum_{j=1}^{n} p_j \leq 1$. Then (3.1) applies. If not all the p_j are equal, then the inequality is strict.*

Proof. The result is trivial with equality if the p_j all share a common value, so we assume at least two different values.

We proceed by induction, Theorem 3.1 providing a basis for $n = 2$. For the inductive step, suppose that (3.1) holds for some $n \geq 2$ and that $\sum_{j=1}^{n+1} p_j \leq 1$. Without loss of generality we may assume that p_{n+1} is the greatest of the values p_j. Since not all the values p_j are equal, we therefore have

$$p_{n+1} > \frac{1}{n} \sum_{j=1}^{n} p_j .$$

This rearranges to give

$$\frac{1}{n} \sum_{j=1}^{n} p_j < \frac{1}{n} \left[p_{n+1} + \frac{n-1}{n+1} \sum_{j=1}^{n+1} p_j \right] .$$

Both sides of this inequality take values in $(0, 1)$.

Also we have

$$\frac{1}{n+1} \sum_{j=1}^{n+1} p_j = \frac{1}{2} \left[\frac{1}{n} \sum_{j=1}^{n} p_j + \frac{1}{n} \left\{ p_{n+1} + \frac{n-1}{n+1} \sum_{j=1}^{n+1} p_j \right\} \right] .$$

Hence applying (3.2) provides

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{2}\left[B\left(\frac{1}{n}\sum_{j=1}^{n}p_j\right) + B\left(\frac{1}{n}\left\{p_{n+1} + \frac{n-1}{n+1}\sum_{j=1}^{n+1}p_j\right\}\right)\right].$$

By the inductive hypothesis

$$B\left(\frac{1}{n}\sum_{j=1}^{n}p_j\right) \geq \frac{1}{n}\sum_{j=1}^{n}B(p_j)$$

and

$$B\left(\frac{1}{n}\left\{p_{n+1} + \frac{n-1}{n+1}\sum_{j=1}^{n+1}p_j\right\}\right) \geq \frac{1}{n}\left[B(p_{n+1}) + (n-1)B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right)\right].$$

Hence

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{2n}\left[\sum_{j=1}^{n+1}B(p_j) + (n-1)B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right)\right].$$

Rearrangement of this inequality yields

$$B\left(\frac{1}{n+1}\sum_{j=1}^{n+1}p_j\right) > \frac{1}{n+1}\sum_{j=1}^{n+1}B(p_j),$$

the desired result. □

Remark 1. *Taken together, relations (2.5) and (3.1) give*

$$n-1+B\left(\sum_{j=1}^{n}p_j\right) < \sum_{j=1}^{n}B(p_j) \le nB\left(\frac{1}{n}\sum_{j=1}^{n}p_j\right), \tag{3.5}$$

the second inequality being strict unless all the values p_j are equal. If $\sum_{j=1}^{n}p_j = 1$, this simplifies to

$$n-1 < \sum_{j=1}^{n}B(p_j) \le nB(n^{-1}), \tag{3.6}$$

since $B(1)=0$.

For $k=0$, (3.5) and (3.6) become (for $m>1$) respectively

$$n-1+\left(1-\left(\sum_{j=1}^{n}p_j\right)^m\right)^m < \sum_{j=1}^{n}(1-p_j^m)^m \le n\left(1-\left(\frac{1}{n}\sum_{j=1}^{n}p_j\right)^m\right)^m$$

and

$$n-1 < \sum_{j=1}^{n}(1-p_j^m)^m \le n(1-n^{-m})^m.$$

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