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THE QUATERNION MATRIX-VALUED YOUNG'S INEQUALITY

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ABSTRACT. In this paper, we prove Young's inequality in quaternion matrices: for any $n \times n$ quaternion matrices A and B, any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, there exists $n \times n$ unitary quaternion matrix Usuch that $U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q$.

Furthermore, there exists unitary quaternion matrix U such that the equality holds if and only if $|B| = |A|^{p-1}$.

Key words and phrases: Quaternion, Matrix, Young's inequality, Real representation.

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1. INTRODUCTION

The two most important classical inequalities probably are the triangle inequality and the arithmetic-geometric mean inequality.

The triangle inequality states that $|\alpha + \beta| \leq |\alpha| + |\beta|$ for any complex numbers α, β.

Thompson [7] extended the classical triangle inequality to $n \times n$ complex matrices: for any $n \times n$ complex matrices A and B, there are $n \times n$ unitary complex matrices U and V such that

$$|A + B| \leq U|A|U^* + V|B|V^*. \tag{1.1}$$

Thompson [6] proved that, the equality in the matrix-valued triangle inequality (1.1) holds if and only if A and B have polar decompositions with a common unitary factor.

Furthermore, Thompson [5] extended the complex matrix-valued triangle inequality (1.1) to the quaternion matrices: for any $n \times n$ quaternion matrices A and B, there are $n \times n$ unitary quaternion matrices U and V such that

$$|A + B| \leq U|A|U^* + V|B|V^*.$$

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The arithmetic-geometric mean inequality is as follows: for any complex numbers α, β,

$$\sqrt{|\alpha\beta|} \leq \frac{1}{2}(|\alpha| + |\beta|);$$

or,

$$|\alpha\beta| \leq \frac{1}{2}(|\alpha|^2 + |\beta|^2),$$

which is a special case of the classical Young's inequality: for any complex numbers α, β, and any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\alpha\beta| \leq \frac{1}{p}|\alpha|^p + \frac{1}{q}|\beta|^q.$$

Bhatia and Kittaneh [2], Ando [1] extended the classical arithmetic-geometric mean inequality and Young's inequality to $n \times n$ complex matrices, respectively. This is Ando's matrix-valued Young's inequality: for any $n \times n$ complex matrices A and B, any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, there is unitary complex matrix U such that

$$U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$

Bhatia and Kittaneh's result is the case of $p = q = 2$, i.e., Young's inequality recovers Bhatia and Kittaneh's arithmetic-geometric-mean inequality, likewise, Ando's matrix version of Young's inequality captures the Bhatia-Kittaneh matricial arithmetic-geometric-mean inequality.

We mention that Erlijman, Farenick and the author [8] proved Young's inequality for compact operators.

This paper extends the Young's inequality to $n \times n$ quaternion matrices and examines the case where equality in the inequality holds.

2. MATRIX-VALUED YOUNG'S INEQUALITY: THE QUATERNION VERSION

We use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ to denote the set of real numbers, the set of complex numbers, and the set of quaternions, respectively.

For any $z \in \mathbb{H}$, we have the unique representation $z = a1 + bi + cj + dk$, where $\{1, i, j, k\}$ is the basis of $\mathbb{H}$. It is well-known that I is the multiplicative identity of $\mathbb{H}$, and $1^2 = i^2 = j^2 = k^2 = -1, \quad ij = k, ki = j, jk = i$, and $ji = -k, ik = -j, kj = -i$.

For each $z = a1 + bi + cj + dk \in \mathbb{H}$, define the conjugate $\bar{z}$ of z by

$$\bar{z} = a1 - bi - cj - dk.$$

Obviously we have $\bar{z}z = z\bar{z} = a^2 + b^2 + c^2 + d^2$. This implies that $\bar{z}z = z\bar{z} = 0$ if and only if $z = 0$. So z is invertible in $\mathbb{H}$ if $z \neq 0$.

We note that as subalgebras of $\mathbb{H}$, the meaning of conjugate in $\mathbb{R}$, or $\mathbb{C}$ is as usual (for any $z \in \mathbb{R}$ we have $\bar{z} = z$).

We can consider $\mathbb{R}$ and $\mathbb{C}$ as real subalgebras of $\mathbb{H}$: $\mathbb{R}$=$\{a1 : a \in \mathbb{R}\}$, and $\mathbb{C}$=$\{a1 + bi : a, b \in \mathbb{R}\}$.

We define the real representation ρ of $\mathbb{H}$, i.e., $\rho : \mathbb{H} \to M_4(\mathbb{R})$ by

$$\rho(z) = \rho(a1 + bi + cj + dk) = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},$$

where $z = a1 + bi + cj + dk \in \mathbb{H}$.

Note that $\rho(\bar{z})$ is the transpose of $\rho(z)$.

From the real representation ρ of $\mathbb{H}$, we define a faithful representation by $\rho_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ as follows:

$$\rho(A) = \rho_n([q_{st}]_{s,t=1}^n) = ([\rho(q_{st})]_{s,t=1}^n)$$

for all matrices $A = [q_{st}]_{s,t=1}^n \in M_n(\mathbb{H})$.

We note that each ρ_n is an injective and homomorphism; and for all $A \in M_n(\mathbb{H})$,

$$\rho_n(A^*) = \rho_n(A)^*.$$

For the set $M_n(\mathbb{F})$ of $n \times n$ matrices with entries from $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we use A^* to denote the conjugate transpose of $A \in M_n(\mathbb{F})$.

We consider $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ as algebras over $\mathbb{R}$, but $M_n(\mathbb{C})$ as a complex algebra.

Definition 2.1. The spectrum $\sigma(A)$ of $A \in M_n(\mathbb{F})$ is a subset of $\mathbb{C}$ that consists of all the roots of the minimal monic annihilating polynomial f of A. We note that if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$, then $f \in \mathbb{R}[x]$; but if $\mathbb{F} = \mathbb{H}$, then $f \in \mathbb{C}[x]$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, then the spectrum $\sigma(A)$ is the set of eigenvalues of A. But if $\mathbb{F} = \mathbb{H}$, then $\sigma(A)$ is the set of eigenvalues of $\rho_n(A)$. A is called Hermitian if $A = A^*$. A is said to be nonnegative definite if A is Hermitian and $\sigma(A)$ are all non-negative real numbers. A is said to be unitary if $A^*A = AA^* = I$, where I is the identity matrix in $M_n(\mathbb{F})$.

If A and B are Hermitian, we define $A \leq B$ or $B \geq A$ if $B - A$ is nonnegative definite.

For any Hermitian matrix A, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ are its eigenvalues, arranged in descending order; where the number of appearances of a particular eigenvalue λ is equal to the dimension of the kernel of $A - \lambda I$ and is known as the geometric multiplicity of λ.

Lemma 2.1 ([1]). *If $A, B \in M_n(\mathbb{C})$, and if $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then there is a unitary $U \in M_n(\mathbb{C})$ such that*

$$U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q,$$

where $|A|$ denotes the nonnegative definite Hermitian matrix

$$|A| = (A^*A)^{\frac{1}{2}}.$$

Lemma 2.2 ([3]). *Let $Q \in M_n(\mathbb{H})$, then Q^*Q is nonnegative definite. Furthermore, if $A \in M_n(\mathbb{H})$ is nonnegative definite, then there are matrices $U, D \in M_n(\mathbb{H})$ such that*

(i) *U is unitary and D is diagonal matrix with nonnegative diagonal entries $d_1, d_2, \dots, d_n$;*
(ii) *$U^*AU = D$;*
(iii) $\sigma(A) = \{d_1, d_2, \dots, d_n\}$;
(iv) *If $\mu \in \sigma(A)$appears t_μ times on the diagonal of D , then the geometric multiplicity of μ as an eigenvalue of $\rho_n(A)$is $4t_\mu$.*

Lemma 2.3. *For any $A, B \in M_n(H)$,*

(i) $\rho_n(|A|) = |\rho_n(A)|$;
(ii) $\rho_n(|A|^p) = |\rho_n(A)|^p$*for any nonnegative definite* p;
(iii) $\rho_n(|AB|) = |\rho_n(A)\rho_n(B)|$.

The meaning of $|A|$ is similar to that in Lemma 2.1, i.e., $|A| = (A^*A)^{\frac{1}{2}}$.

Proof. (i) Note that $\rho_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ is a homomorphism, if $X \in M_n(\mathbb{H})$ is nonnegative definite, then there is a $Y \in M_n(\mathbb{H})$ such that $X = YY^*$, so

$$\rho_n(X) = \rho_n(Y^*Y) = \rho_n(Y^*) \cdot \rho_n(Y) = \rho_n(Y)^* \cdot \rho_n(Y) = |\rho_n(Y)|^2_,$$

which means that $\rho_n(X)$ is also nonnegative definite. Hence, for any $X \in M_n(\mathbb{H})$ we have (since ρ_n is a homomorphism),

$$\left(\rho_n(|X|)^{\frac{1}{2}}\right)^2 = \rho_n(|X|) = \rho_n\left(|X|^{\frac{1}{2}} \cdot |X|^{\frac{1}{2}}\right) = \left(\rho_n\left(|X|^{\frac{1}{2}}\right)\right)^2.$$

So $\rho_n(|X|)^{\frac{1}{2}} = \rho_n\left(|X|^{\frac{1}{2}}\right)$. Therefore

$$\rho_n(|A|) = (\rho_n(A^*A))^{\frac{1}{2}} = (\rho_n(A^*)\rho_n(A))^{\frac{1}{2}} = |\rho_n(A)|.$$

We get (i).
(ii) For any nonnegative definite p,

$$\rho_n(|A|^p) = (\rho_n(|A|))^p = |\rho_n(A)|^p,$$

the first equality is because $\rho_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ is a homomorphism, and the second equality is from (i).
(iii) Similar to (ii) we have

$$\rho_n(|AB|) = |\rho_n(AB)| = |\rho_n(A)\rho_n(B)|.$$

The proof is complete. □

The following Theorem 2.4 is one of our main results.

Theorem 2.4. *For any* $A, B \in M_n(\mathbb{H})$*, any* $p, q \in (1, \infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, there is a unitary* $U \in M_n(\mathbb{H})$*, such that*

$$U|AB^*|U^* \le \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$

Proof. By Lemma 2.3 $\rho_n(|AB^*|) = |\rho_n(A)\rho_n(B)^*|$, and

$$\rho_n\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right) = \frac{1}{p}|\rho_n(A)|^p + \frac{1}{q}|\rho_n(B)|^q.$$

Because real $n \times n$ matrices $|\rho_n(A)\rho_n(B)^*|$ and $\frac{1}{p}|\rho_n(A)|^p + \frac{1}{q}|\rho_n(B)|^q$ are nonnegative definite, from Linear Algebra there are $n \times n$ unitary matrices $V, W \in M_n(\mathbb{C})$ such that

$$V|\rho_n(A)\rho_n(B)^*|V^* = C \quad \text{and} \quad W\left(\frac{1}{p}|\rho_n(A)|^p + \frac{1}{q}|\rho_n(B)|^q\right)W^* = D,$$

where C and D are diagonal matrices in $M_{4n}(\mathbb{R})$.

Thus from Lemma 2.2(iv) one has

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_n \quad \text{and} \quad D = D_1 \oplus D_2 \oplus \cdots \oplus D_n$$

with $C_s = diag\{c_s, c_s, \ldots, c_s\}$ and $D_s = diag\{d_s, d_s, \ldots, d_s\}$, where c_s and d_s are nonnegative real numbers, $s = 1, 2, \ldots, n$. By Lemma 2.2 (iii) we have

$$\sigma(|AB^*|) = \{c_1, c_2, \ldots, c_s\}$$

and

$$\sigma\left(\frac{1}{p}|\rho_n(A)|^p + \frac{1}{q}|\rho_n(B)|^q\right) = \{d_1, d_2, \ldots, d_n\}.$$

Furthermore, Lemma 2.2 implies that

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_n \le D = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

Hence the equation above and Lemma 2.3 yield that

$$diag\{c_1, c_2, \ldots, c_n\} \le diag\{d_1, d_2, \ldots, d_n\}.$$

Thus from Lemma 2.2 (i) (ii) (iii) there are unitary matrices $U_1, U_2 \in M_n(\mathbb{H})$ such that

$$U_1|AB^*|U_1^* \leq U_2 \left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right) U_2^*,$$

then there is a unitary matrix $U \in M_n(\mathbb{H})$ for which

$$U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$

The proof is complete. □

3. THE CASE OF EQUALITY

Hirzallah and Kittaneh [4] proved a result as follows.

Lemma 3.1. *Let $A, B \in M_n(\mathbb{C})$ be nonnegative definite. If $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and if there exists unitary $U \in M_n(\mathbb{C})$ such that*

$$U|AB|U^* = \frac{1}{p}A^p + \frac{1}{q}B^q$$

then $B = A^{p-1}$.

We have the following result.

Theorem 3.2. *For any $A, B \in M_n(\mathbb{H})$, any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, there is a unitary $U \in M_n(\mathbb{H})$ such that*

$$U|AB^*|U^* = \frac{1}{p}|A|^p + \frac{1}{q}|B|^q \tag{3.1}$$

if and only if $|B| = |A|^{p-1}$.

Proof. The sufficiency. In fact, if $|B| = |A|^{p-1}$ then

$$|\rho_n(B)| = \rho_n(|B|) = \rho_n(|A|^{p-1}) = |\rho_n(A)|^{p-1}.$$

Write $X = \rho_n(A), Y = \rho_n(B)$.

Suppose $X = V|X|, Y = W|Y|$ are the polar decomposition of X, Y respectively, where V, W are $4n \times 4n$ unitary complex matrices. Then from (3.1) we have

$$|XY^*| = W||X||Y||W^* = W|X|^pW^*.$$

Simply computation yields

$$\frac{1}{p}|X|^p + \frac{1}{q}|Y|^q = |X|^p.$$

So

$$W^*|XY^*|W = \frac{1}{p}|X|^p + \frac{1}{q}|Y|^q.$$

Since W is a unitary, using the notations in Theorem 2.4, this implies

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_n = D = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

Hence Lemma 2.2 yields that

$$diag\{c_1, c_2, \ldots, c_n\} = diag\{d_1, d_2, \ldots, d_n\}.$$

Again, by Lemma 2.2, there is a unitary $U \in M_n(\mathbb{H})$ such that

$$U|AB^*|U^* = \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$

The necessity. Assume there exists unitary $U \in M_n(\mathbb{H})$ such that (3.1) holds, i.e.

$$U|AB^*|U^* = \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$

Then

$$\rho_n(U|AB^*|U^*) = \rho_n\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right).$$

Writing $X = \rho_n(A), Y = \rho_n(B)$, and $T = \rho_n(U)$, one gets

$$T|XY^*|T^* = \frac{1}{p}|X|^p + \frac{1}{q}|Y|^q.$$

This and Lemma 3.1 imply that

$$|Y| = (|X|^p)^{\frac{1}{q}} = |X|,$$

which means

$$\rho_n(|B|) = \rho_n(|A|)^{p-1} = \rho_n(|A|^{p-1}).$$

Therefore (note that $\rho_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ is a faithful representation)

$$|B| = |A|^{p-1}.$$

This completes the proof. □

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