Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 3, Article 50, 2003

CONVOLUTION INEQUALITIES AND APPLICATIONS

SABUROU SAITOH, VU KIM TUAN, AND MASAHIRO YAMAMOTO

DEPARTMENT OF MATHEMATICS,
FACULTY OF ENGINEERING,
GUNMA UNIVERSITY,
KIRYU 376-8515 JAPAN
ssaitoh@math.sci.gunma-u.ac.jp

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
FACULTY OF SCIENCE,
KUWAIT UNIVERSITY,
P.O. BOX 5969,
SAFAT 13060 KUWAIT
vu@mcs.sci.kuniv.edu.kw

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
THE UNIVERSITY OF TOKYO,
3-8-1 KOMABA MEGURO
TOKYO 153-8914 JAPAN
myama@ms.u-tokyo.ac.jp

Received 09 December, 2002; accepted 15 April, 2003
Communicated by L.-E. Persson

ABSTRACT. We introduce various convolution inequalities obtained recently and at the same time, we give new type of reverse convolution inequalities and their important applications to inverse source problems. We consider the inverse problem of determining $f(t)$, $0 < t < T$, in the heat source of the heat equation $\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x)$, $x \in R^n$, $t > 0$ from the observation $u(x_0, t)$, $0 < t < T$, at a remote point x_0 away from the support of φ. Under an a priori assumption that f changes the signs at most N-times, we give a conditional stability of Hölder type, as an example of applications.

Key words and phrases: Convolution, Heat source, Weighted convolution inequalities, Young's inequality, Hölder's inequality, Reverse Hölder's inequality, Green's function, Stability in inverse problems, Volterra's equation, Conditional stability of Hölder type, Analytic semigroup, Interpolation inequality, Sobolev inequality.

2000 *Mathematics Subject Classification.* Primary 44A35; Secondary 26D20.

ISSN (electronic): 1443-5756

The authors wish to express our deep thanks for the referee for his or her careful reading of the manuscript. In particular, he or she found misprints in Theorem 3.1 which also appeared in Theorem 3.1 in [17].

138-02

1. INTRODUCTION

For the Fourier convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi,$$

the Young's inequality

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q, \quad f \in L_p(R), g \in L_q(R), \tag{1.1}$$

$$r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0),$$

is fundamental. Note, however, that for the typical case of $f, g \in L_2(R)$, the inequality does not hold. In a series of papers [12]–[15] (see also [5]) we obtained the following weighted $L_p(p > 1)$ norm inequality for convolution:

Proposition 1.1. *(*[15]*) For two non-vanishing functions $\rho_j \in L_1(R)(j = 1, 2)$, the $L_p(p > 1)$ weighted convolution inequality*

$$\left\| ((F_1\rho_1) * (F_2\rho_2)) \, (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \leq \|F_1\|_{L_p(R,|\rho_1|)} \, \|F_2\|_{L_p(R,|\rho_2|)} \tag{1.2}$$

holds for $F_j \in L_p(R, |\rho_j|)(j = 1, 2)$. Equality holds here if and only if

$$F_j(x) = C_j e^{\alpha x}, \tag{1.3}$$

where α is a constant such that $e^{\alpha x} \in L_p(R, |\rho_j|)(j = 1, 2)$ (otherwise, C_1 or $C_2 = 0$). Here

$$\|F\|_{L_p(R,|\rho|)} = \left\{ \int_{-\infty}^{\infty} |F(x)|^p |\rho(x)| \, dx \right\}^{\frac{1}{p}}.$$

Unlike Young's inequality, inequality (1.2) holds also in the case $p = 2$.

Note that the proof of inequality (1.2) in Proposition 1.1 is direct and fairly elementary. The proof will be done in three lines. Indeed, we use only Hölder's inequality and Fubini's theorem for exchanging the orders of integrals for the proof. So, for various type convolutions, we can also obtain similar type convolution inequalities, see [18] for various convolutions. However, to determine the case that equality in (1.2) holds needs very delicate arguments. See [5] for the details.

In many cases of interest, the convolution is given in the form

$$\rho_2(x) \equiv 1, \quad F_2(x) = G(x), \tag{1.4}$$

where $G(x - \xi)$ is some Green's function. Then the inequality (1.2) takes the form

$$\|(F\rho) * G\|_p \leq \|\rho\|_p^{1-\frac{1}{p}} \, \|G\|_p \, \|F\|_{L_p(\mathbb{R},|\rho|)}, \tag{1.5}$$

where ρ, F, and G are such that the right hand side of (1.5) is finite.

Inequality (1.5) enables us to estimate the output function

$$\int_{-\infty}^{\infty} F(\xi)\rho(\xi)G(x-\xi)\,d\xi \tag{1.6}$$

in terms of the input function F in the related differential equation. For various applications, see [15]. We are also interested in the reverse type inequality of (1.5), namely, we wish to estimate the input function F by means of the output (1.6). This kind of estimate is important in inverse problems. One estimate is obtained by using the following famous reverse Hölder inequality

Proposition 1.2. *([19], see also [10, pp. 125-126]). For two positive functions f and g satisfying*

$$0 < m \le \frac{f}{g} \le M < \infty \tag{1.7}$$

on the set X, and for $p, q > 1,\ p^{-1} + q^{-1} = 1$,

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \le A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{1.8}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1 - t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1 - t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

Then, by using Proposition 1.2 we obtain, as in the proof of Proposition 1.1, the following:

Proposition 1.3. *([16]). Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \le F_1(x) \le M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \le F_2(x) \le M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x \in R. \tag{1.9}$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the reverse L_p–weighted convolution inequality

$$\left\{A_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\right\}^{-1} \|F_1\|_{L_p(R,\rho_1)} \|F_2\|_{L_p(R,\rho_2)} \le \left\| \left((F_1\rho_1) * (F_2\rho_2)\right)(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p. \tag{1.10}$$

Inequality (1.10) and others should be understood in the sense that if the right hand side is finite, then so is the left hand side, and in this case the inequality holds.

In formula (1.10) replacing ρ_2 by 1, and $F_2(x-\xi)$ by $G(x-\xi)$, and taking integration with respect to x from c to d we arrive at the following inequality

$$\begin{aligned} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \left(\int_{-\infty}^{\infty} \rho(\xi)\, d\xi\right)^{p-1} \int_{-\infty}^{\infty} F^p(\xi)\, \rho(\xi)\, d\xi \int_{c-\xi}^{d-\xi} G^p(x) dx \\ \le \int_c^d \left(\int_{-\infty}^{\infty} F(\xi)\rho(\xi) G(x-\xi) d\xi\right)^p dx \end{aligned} \tag{1.11}$$

if the positive continuous functions ρ, F, and G satisfy

$$0 < m^{\frac{1}{p}} \le F(\xi) G(x-\xi) \le M^{\frac{1}{p}}, \quad x \in [c,d], \quad \xi \in R. \tag{1.12}$$

Inequality (1.11) is especially important when $G(x-\xi)$ is a Green function. We gave various concrete applications in [16] from the viewpoint of stability in inverse problems.

2. REMARKS FOR REVERSE HÖLDER INEQUALITIES

In connection with Proposition 1.2 which gives Proposition 1.3, Izumino and Tominaga [8] consider the upper bound of

$$\left(\sum a_k^p\right)^{\frac{1}{p}} \left(\sum b_k^q\right)^{\frac{1}{q}} - \lambda \sum a_k b_k$$

for $\lambda > 0$, for $p, q > 1$ satisfying $1/p + 1/q = 1$ and for positive numbers $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$, in detail. In their different approach, they showed that the constant $A_{p,q}(t)$ in Proposition 1.2 is best possible in a sense. Note that the proof of Proposition 1.2 is quite involved. In connection with Proposition 1.2 we note that the following version whose proof is surprisingly simple

Theorem 2.1. *([17]). In Proposition 1.2, replacing f and g by f^p and g^q, respectively, we obtain the reverse Hölder type inequality*

$$\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \left(\int_X g^q d\mu\right)^{\frac{1}{q}} \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}} \int_X fg d\mu. \tag{2.1}$$

Proof. Since $f^p/g^q \leq M$, $g \geq M^{-\frac{1}{q}} f^{\frac{p}{q}}$. Therefore

$$fg \geq M^{-\frac{1}{q}} f^{1+\frac{p}{q}} = M^{-\frac{1}{q}} f^p$$

and so,

$$\left\{\int f^p d\mu\right\}^{\frac{1}{p}} \leq M^{\frac{1}{pq}} \left\{\int fg d\mu\right\}^{\frac{1}{p}}. \tag{2.2}$$

On the other hand, since $m \leq f^p/g^q$, $f \geq m^{1/p} g^{q/p}$. Hence

$$\int fg d\mu \geq \int m^{\frac{1}{p}} g^{1+\frac{q}{p}} d\mu = m^{\frac{1}{p}} \int g^q d\mu,$$

and so,

$$\left\{\int fg d\mu\right\}^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left\{\int g^q d\mu\right\}^{\frac{1}{q}}.$$

Combining with (2.2), we have the desired inequality

$$\begin{aligned}\left\{\int f^p d\mu\right\}^{\frac{1}{p}} \left\{\int g^q d\mu\right\}^{\frac{1}{q}} &\leq M^{\frac{1}{pq}} \left\{\int fg d\mu\right\}^{\frac{1}{p}} m^{\frac{-1}{pq}} \left\{\int fg d\mu\right\}^{\frac{1}{q}} \\ &= \left(\frac{m}{M}\right)^{\frac{-1}{pq}} \int fg d\mu.\end{aligned}$$

□

Remark 2.2. In a private communication, Professor Lars-Erik Persson pointed out the following very interesting result that

$$A_{p,q}(t) < t^{\frac{-1}{pq}} \tag{LEP}$$

which implies that Theorem 2.1 can be derived from Proposition 1.2, directly. Its proof is noted as follows:

By using an obvious estimate and Hölder's inequality we find that

$$\begin{aligned}1 - t &= \int_t^1 1 dt \\ &= \int_t^1 u^{1/p^2} u^{1/q^2} u^{1/p-1/p^2} u^{1/q-1/q^2} \frac{du}{u} \\ &< \int_t^1 u^{1/p^2} u^{1/q^2} \frac{du}{u} \\ &\leq \left(\int_t^1 u^{1/p} \frac{du}{u}\right)^{\frac{1}{p}} \left(\int_t^1 u^{1/q} \frac{du}{u}\right)^{\frac{1}{q}} \\ &= [p(1-t^{1/p}]^{1/p} [q(1-t^{1/q}]^{1/q},\end{aligned}$$

which implies (LEP).

3. NEW REVERSE CONVOLUTION INEQUALITIES

In reverse convolution inequality (1.10), similar type inequalities for $m_1 = m_2 = 0$ are also important as we see from our example in Section 4. For these, we obtain a reverse convolution inequality of new type.

Theorem 3.1. *Let* $p \geq 1, \delta > 0$, $0 \leq \alpha < T$, *and* $f, g \in L_\infty(0, T+\delta)$ *satisfy*

$$0 \leq f, g \leq M < \infty, \qquad 0 < t < T+\delta. \tag{3.1}$$

Then

$$\|f\|_{L_p(\alpha,T)} \|g\|_{L_p(0,\delta)} \leq M^{\frac{2p-2}{p}} \left(\int_\alpha^{T+\delta} \left(\int_\alpha^t f(s) g(t-s) ds \right) dt \right)^{\frac{1}{p}}. \tag{3.2}$$

In particular, for

$$(f * g)(t) = \int_0^t f(t-s) g(s) ds, \qquad 0 < t < T+\delta$$

and for $\alpha = 0$, *we have*

$$\|f\|_{L_p(0,T)} \|g\|_{L_p(0,\delta)} \leq M^{\frac{2p-2}{p}} \|f * g\|_{L_1(0,T+\delta)}^{\frac{1}{p}}.$$

Proof. Since $0 \leq f, g \leq M$ for $0 \leq t \leq T$, we have

$$\begin{aligned} \int_\alpha^t f(s)^p g(t-s)^p ds &= \int_\alpha^t f(s)^{p-1} g(t-s)^{p-1} f(s) g(t-s) ds \\ &\leq M^{2p-2} \int_\alpha^t f(s) g(t-s) ds. \end{aligned} \tag{3.3}$$

Hence

$$\int_\alpha^{T+\delta} \left(\int_\alpha^t f(s)^p g(t-s)^p ds \right) dt \leq M^{2p-2} \int_\alpha^{T+\delta} \left(\int_\alpha^t f(s) g(t-s) ds \right) dt.$$

On the other hand, we have

$$\begin{aligned} \int_\alpha^{T+\delta} \left(\int_\alpha^t f(s)^p g(t-s)^p ds \right) dt &= \int_\alpha^{T+\delta} \left(\int_s^{T+\delta} g(t-s)^p dt \right) f(s)^p ds \\ &= \int_\alpha^{T+\delta} \left(\int_0^{T+\delta-s} g(\eta)^p d\eta \right) f(s)^p ds \\ &\geq \int_\alpha^{T} \left(\int_0^{T+\delta-s} g(\eta)^p d\eta \right) f(s)^p ds \\ &\geq \int_\alpha^{T} \left(\int_0^{\delta} g(\eta)^p d\eta \right) f(s)^p ds \\ &= \|f\|_{L_p(\alpha,T)}^p \|g\|_{L_p(0,\delta)}^p. \end{aligned}$$

Thus the proof of Theorem 3.1 is complete. □

4. APPLICATIONS TO INVERSE SOURCE HEAT PROBLEMS

We consider the heat equation with a heat source:

$$\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x), \quad x \in \mathbb{R}^n,\ t > 0, \tag{4.1}$$

$$u(x,0) = 0, \qquad x \in \mathbb{R}^n. \tag{4.2}$$

We assume that φ is a given function and satisfies $\varphi \geq 0, \quad \not\equiv 0 \quad \text{in} \quad \mathbb{R}^n$, φ has compact support,

$$\begin{cases} \varphi \in C^\infty(\mathbb{R}^n), & \text{if} \quad n \geq 4 \quad \text{and} \\ \varphi \in L_2(\mathbb{R}^n), & \text{if} \quad n \leq 3. \end{cases} \tag{4.3}$$

Our problem is to derive a conditional stability in the determination of $f(t)$, $0 < t < T$, from the observation

$$u(x_0, t), \qquad 0 < t < T, \tag{4.4}$$

where $x_0 \notin \text{supp } \varphi$.

We are interested only in the case of $x_0 \notin \operatorname{supp} \varphi$, because in the case where x_0 is in the interior of $\operatorname{supp} \varphi$, the problem can be reduced to a Volterra integral equation of the second kind by differentiation in t formula (4.8) stated below. Moreover $x_0 \notin \operatorname{supp} \varphi$ means that our observation (4.4) is done far from the set where the actual process is occurring, and the design of the observation point is easy.

Let

$$K(x, t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.5}$$

Then the solution u to (4.1) and (4.2) is represented by

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} K(x - y, t - s) f(s) \varphi(y) dy ds, \quad x \in \mathbb{R}^n,\ t > 0, \tag{4.6}$$

(see e.g. Friedman [6]). Therefore, setting

$$\mu_{x_0}(t) = \int_{\mathbb{R}^n} K(x_0 - y, t) \varphi(y) dy, \qquad t > 0, \tag{4.7}$$

we have

$$u(x_0, t) \equiv h_{x_0}(t) = \int_0^t \mu_{x_0}(t - s) f(s) ds, \quad 0 < t < T, \tag{4.8}$$

which is a Volterra integral equation of the first kind with respect to f. Since

$$\lim_{t \downarrow 0} \frac{d^k \mu_{x_0}}{dt^k}(t) = (\Delta^k \varphi)(x_0) = 0, \quad k \in \mathbb{N} \cup \{0\}$$

by $x_0 \notin \operatorname{supp} \varphi$ (e.g. [6]), the equation (4.8) cannot be reduced to a Volterra equation of the second kind by differentiating in t. Hence, even though, for any $m \in \mathbb{N}$, we take the C^m-norms for data h, the equation (4.8) is ill-posed, and we cannot expect a better stability such as of Hölder type under suitable a priori boundedness.

In Cannon and Esteva [3], an estimate of logarithmic type is proved: let $n = 1$ and $\varphi = \varphi(x)$ be the characteristic function of an interval $(a, b) \subset \mathbb{R}$. Set

$$\mathcal{V}_M = \left\{ f \in C^2[0, \infty); f(0) = 0, \quad \left\| \frac{df}{dt} \right\|_{C[0,\infty)}, \left\| \frac{d^2 f}{dt^2} \right\|_{C[0,\infty)} \leq M \right\}. \tag{4.9}$$

Let $x_0 \notin (a, b)$. Then, for $T > 0$, there exists a constant $C = C(M, a, b, x_0) > 0$ such that

$$|f(t)| \leq \frac{C}{|\log \|u(x_0, \cdot)\|_{L_2(0,\infty)}|^2}, \quad 0 \leq t \leq T, \tag{4.10}$$

for all $f \in \mathcal{V}_M$. The stability rate is logarithmic and worse than any rate of Hölder type: $\|u(x_0, \cdot)\|^\alpha_{L_2(0,\infty)}$ for any $\alpha > 0$. For (4.9), the condition $f \in \mathcal{V}_M$ prescribes a priori information and (4.9) is called conditional stability within the admissible set $\mathcal{V}_M$. The rate of conditional stability heavily depends on the choice of admissible sets and an observation point x_0. As for other inverse problems for the heat equation, we can refer to Cannon [2], Cannon and Esteva [4], Isakov [7] and the references therein.

For fixed $M > 0$ and $N \in \mathbb{N}$ let

$$\mathcal{U} = \{f \in C[0,T]; \|f\|_{C[0,T]} \le M, f \text{ changes the signs at most } N\text{-times}\}. \tag{4.11}$$

We take $\mathcal{U}$ as an admissible set of unknowns f. Then, within $\mathcal{U}$, we can show an improved conditional stability of Hölder type:

Theorem 4.1. *Let φ satisfy (4.3), and $x_0 \notin \operatorname{supp}\varphi$. We set*

$$p > \begin{cases} \frac{4}{4-n}, & n \le 3, \\ 1, & n \ge 4. \end{cases} \tag{4.12}$$

Then, for an arbitrarily given $\delta > 0$, there exists a constant $C = C(x_0, \varphi, T, p, \delta, \mathcal{U}) > 0$ such that

$$\|f\|_{L_p(0,T)} \le C\|u(x_0,\cdot)\|_{L_1(0,T+\delta)}^{\frac{1}{p^N}} \tag{4.13}$$

for any $f \in \mathcal{U}$.

We will see that $\lim_{\delta\to 0} C = \infty$ and, in order to estimate f over the time interval $(0,T)$, we have to observe $u(x_0,\cdot)$ over a longer time interval $(0, T+\delta)$.

For a quite long proof of Theorem 4.1 and for the following remarks, see [17].

Remark 4.2. In the case of $n \ge 4$, we can relax the regularity of φ to $H^\alpha(\mathbb{R}^n)$ with some $\alpha > 0$. In the case of $n \le 3$, if we assume that $\varphi \in C^\infty(\mathbb{R}^n)$ in (4.3), then in Theorem 4.1 we can take any $p > 1$.

Remark 4.3. As a subset of $\mathcal{U}$, we can take, for example,

$$\mathcal{P}_N = \{f; f \text{ is a polynomial whose order is at most } N \text{ and } \|f\|_{C[0,T]} \le M\}.$$

The condition $f \in \mathcal{U}$ is quite restrictive at the expense of the practically reasonable estimate of Hölder type.

Remark 4.4. The a priori boundedness $\|f\|_{C[0,T]} \le M$ is necessary for the stability. See [17] for a counter example.

Remark 4.5. For our stability, the finiteness of changes of signs is essential. In fact, we take

$$f_n(t) = \cos nt, \qquad 0 \le t \le T, \; n \in \mathbb{N}. \tag{4.14}$$

Then f_n oscillates very frequently and we cannot take any finite partition of $(0,T)$ where the condition on signs in (4.11) holds true. We note that we can take $M = 1$, that is, $\|f_n\|_{C[0,T]} \le 1$ for $n \in \mathbb{N}$. We denote the solution to (4.1) – (4.2) for $f = f_n$ by $u_n(x,t)$. Then, we see that any stability cannot hold for f_n, $n \in \mathbb{N}$. See [17] for the proof.

REFERENCES

[1] R.A. ADAMS, *Sobolev Spaces*, Academic Press, New York (1975).

[2] J.R. CANNON, *The One-dimensional Heat Equation*, Addison-Wesley Reading, Massachusetts (1984).

[3] J.R. CANNON AND S.P. ESTEVA, An inverse problem for the heat equation, *Inverse Problems*, **2** (1986), 395–403.

[4] J.R. CANNON AND S.P. ESTEVA, Uniqueness and stability of 3D heat sources, *Inverse Problems*, **7** (1991), 57–62.

[5] M. CWICKEL AND R. KERMAN, On a convolution inequality of Saitoh, *Proc. Amer. Math. Soc.*, **124** (1996), 773–777.

[6] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Krieger Malabar, Florida (1983).

[7] V. ISAKOV, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin (1998).

[8] S. IZUMINO AND M. TOMINAGA, Estimations in Hölder type inequalities, *Math. Inequal. Appl.*, **4** (2001), 163–187.

[9] J.-L. LIONS AND E. MAGENES, *Non-homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin (1972).

[10] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[11] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin (1983).

[12] S. SAITOH, A fundamental inequality in the convolution of L_2 functions on the half line, *Proc. Amer. Math. Soc.*, **91** (1984), 285–286.

[13] S. SAITOH, Inequalities in the most simple Sobolev space and convolutions of L_2 functions with weights, *Proc. Amer. Math. Soc.*, **118** (1993), 515–520.

[14] S. SAITOH, Various operators in Hilbert spaces introduced by transform, *International. J. Appl. Math.*, **1** (1999), 111–126.

[15] S. SAITOH, Weighted L_p-norm inequalities in convolutions, *Survey on Classical Inequalities*, 225–234 (2000), Kluwer Academic Publishers, The Netherlands.

[16] S. SAITOH, V.K. TUAN AND M. YAMAMOTO, Reverse weighted L_p-norm inequalities in convolutions and stability in inverse problems, *J. of Inequal. Pure and Appl. Math.*, **1**(1) (2000), Article 7. [ONLINE: `http://jipam.vu.edu.au/v1n1/018_99.html`]

[17] S. SAITOH, V.K. TUAN AND M. YAMAMOTO, Reverse convolution inequalities and applications to inverse heat source problems, *J. of Inequal. Pure and Appl. Math.*, **3**(5) (2003), Article 80. [ONLINE: `http://jipam.vu.edu.au/v3n5/029_02.html`]

[18] H.M. SRIVASTAVA AND R.G. BUSCHMAN, *Theory and Applications of Convolution Integral Equations*, Kluwer Academic Publishers, The Netherlands (1992).

[19] Xiao-Hua, L., On the inverse of Hölder inequality, *Math. Practice and Theory*, **1** (1990), 84–88.