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SOME NEW INEQUALITIES FOR GAMMA AND POLYGAMMA FUNCTIONS

NECDET BATIR

DEPARTMENT OF MATHEMATICS
FACULTY OF ARTS AND SCIENCE
YUZUNCU YIL UNIVERSITY, 65080, VAN, TURKEY
necdet_batir@hotmail.com

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ABSTRACT. In this paper we derive some new inequalities involving the gamma function Γ, polygamma functions $\psi = \Gamma'/\Gamma$ and ψ'. We also obtained two new sequences converging to Euler-Mascheroni constant γ very quickly.

Key words and phrases: Digamma function, psi function, polygamma function, gamma function, inequalities, Euler's constant and completely monotonicity.

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1. INTRODUCTION

For $x > 0$ let $\Gamma(x)$ and $\psi(x)$ denote the Euler's gamma function and psi (digamma) function, defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$$

and

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

respectively. The derivatives ψ',ψ'',ψ''', ... are known as polygamma functions. A good reference for these functions is [8].

The gamma and polygamma functions play a central role in the theory of special functions and they are closely related to many of them such as the Riemann zeta-function, the Clausen integral etc. They have many applications in mathematical physics and statistics. In the recent past, several articles have appeared providing various inequalities for gamma and polygamma functions; see ([2], [3], [4], [5], [6], [7], [10], [12], [14]).

It is the aim of this paper to continue these investigations and to present some new inequalities for the gamma function and some polygamma functions. Our results also lead to two new sequences converging to the Euler- Mascheroni constant γ very quickly. Throughout this paper,

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$c = 1.461632144968362$ denotes the only positive root of the ψ-function (see [1, p. 259; 6.3.19]).

Before establishing our main result we need to prove two lemmas.

Lemma 1.1. *For* $x > 0$, $[\psi'(x)]^2 + \psi''(x) > 0$.

Proof. To prove the lemma we define the function $f(x)$ by

$$f(x) = [\psi'(x)]^2 + \psi''(x), \quad x > 0.$$

Since $\lim_{x\to\infty} f(x) = 0$ in order to show that $f(x) > 0$, it is sufficient to show that $f(x) - f(x+1) > 0$ for $x > 0$. Now

$$f(x) - f(x+1) = [\psi'(x)]^2 + \psi''(x) - [\psi'(x+1)]^2 - \psi''(x+1). \tag{1.1}$$

From the well-known difference equation $\Gamma(x+1) = x\Gamma(x)$ [8, (1.1.6)] it follows easily that

$$\psi(x+1) - \psi(x) = \frac{1}{x}. \tag{1.2}$$

Differentiating both sides of this equality, we get

$$\psi'(x+1) - \psi'(x) = -\frac{1}{x^2}. \tag{1.3}$$

Thus, (1.1) can be written as

$$f(x) - f(x+1) = \frac{2}{x^2}\left(\psi'(x) - \frac{1}{x} - \frac{1}{2x^2}\right).$$

By [12, p. 2670], we have

$$\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} > 0, \tag{1.4}$$

concluding $f(x) - f(x+1) > 0$ for $x > 0$. This proves Lemma 1.1 □

Lemma 1.2. *For* $x > 0$, $\psi'(x)\, e^{\psi(x)} < 1$.

Proof. By Lemma 1.1 we have

$$\frac{d}{dx}(\psi(x) + \ln \psi'(x)) > 0, \quad x > 0.$$

Thus the function $\psi(x) + \ln \psi'(x)$ is strictly increasing on $(0, \infty)$. By [7] for $x > 0$ we have

$$\log x - \frac{1}{x} < \psi(x) < \log x - \frac{1}{2x}.$$

This gives

$$x\psi'(x)\, e^{-1/x} < \psi'(x) e^{\psi(x)} < x\psi'(x)\, e^{-1/2x}. \tag{1.5}$$

Using the asymptotic representation [1, p. 260; 6.4.12]

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2\,z^2} + \frac{1}{6\,z^3} - \frac{1}{30\,z^5} + \cdots \quad (\text{as } z \to \infty,\ |\arg z| < \pi),$$

which will be used only for real z's in this paper, we get

$$\lim_{x\to\infty} x\psi'(x) = 1.$$

Hence, by (1.5), we find that

$$\lim_{x\to\infty} \psi'(x)\, e^{\psi(x)} = 1. \tag{1.6}$$

or

$$\lim_{x\to\infty}[\log\psi'(x)+\ \psi(x)]=0. \tag{1.7}$$

Now the proof follows from the monotonicity of $\psi(x)+\ln(\psi'(x))$ and the limit in (1.7) $\square$

2. MAIN RESULTS

The main result of this paper is the following theorem.

Theorem 2.1. *The functions ψ, ψ' and Γ satisfy the following inequalities:*

a) *for $x \geq 1$*

$$\psi(x) \leq \log\left(x-1+e^{-\gamma}\right),$$

and for $x > 0.5$

$$\psi(x) > \log(x-0.5).$$

Both of the constants $1-e^{-\gamma}=0.438540516$ and 0.5 are best possible with γ is Euler-Mascheroni constant.

b) *For $x > 0$*

$$-\log 2-\log\left(e^{1/x}-1\right) < \psi(x) < -\log(e^{1/x}-1).$$

c) *For $x \geq 2$*

$$\psi(x) > \log(\pi^2/6)-\gamma-\log(e^{1/x}-1).$$

d) *For $x \geq 1$*

$$\psi'(x) \geq \frac{\pi^2}{6e^{\gamma}}e^{-\psi(x)}.$$

e) *For $x > 0$ and $h > 0$*

$$\log(1+h\psi'(x)) < \psi(x+h)-\psi(x) < -\log\left(\ 1-h\psi'(x+h)\right)$$

f) *For $x > 0$*

$$1+\frac{1}{x^2}-e^{-1/x} < \psi'(x) < \frac{1}{x^2}-1+e^{1/(x+1)}.$$

g) *For $x > 1$*

$$\log x-\psi(x) < \frac{1}{2}\,\psi'(x)$$

h) *For $x > 1$*

$$\log x-\psi(x) > (c-1)\,\psi'(x+1/2)$$

where $c=1.461632144968362$ is the only positive root of $\psi-$ function (see [1, p. 259; 6.3.19]*).*

i) *For $x \geq 1/2$*

$$\Gamma(x+1) \geq \Gamma(c)(x+0.5)^{x+0.5}e^{-x+0.5}.$$

j) *For $x \geq c-1=0.461632144968362$*

$$\Gamma(x+1) \leq \Gamma(c)(x+2-c)^{6(x\,+\,2\,-\,c)\;e^{\gamma}/\pi^2}e^{6(-x\,-\,1+c)\;e^{\gamma}/\pi^2}.$$

Here $\Gamma(c)=0.885603194410889$; see [1, p. 259;6.3.9].

Proof. Applying the mean value theorem to the function $\log\Gamma(x)$ on $[u, u+1]$ with $u > 0$, there exists a θ depending on u such that for all $u \geq 0$, $0 \leq \theta = \theta(u) < 1$ and

$$\log\Gamma(u+1) - \log\Gamma(u) = \psi(u + \theta\,(u)).$$

Using the well-known difference equation $\Gamma(u+1) = u\,\Gamma(u)$, this becomes for $u > 0$

$$\psi(u + \theta\,(u)) = \log u. \tag{2.1}$$

First, we are going to show that the function $\theta(u)$ has the following four properties:

P_1 : θ is strictly increasing on $(0, \infty)$.

P_2 : $\lim\limits_{u\to\infty} \theta\,(u) = \frac{1}{2}$.

P_3 : θ' is strictly decreasing on $(0, \infty)$.

P_4 : $\lim\limits_{u\to\infty} \theta'(u) = 0$.

Put $u = e^{\psi(t)}$ with $t > 0$ in (2.1) to obtain

$$\psi(e^{\psi(t)} + \theta(e^{\psi(t)})) = \psi(t).$$

Since the mapping $t \to \psi(t)$ from $(0, \infty)$ to $(-\infty, \infty)$ is bijective, we find that

$$\theta(e^{\psi(t)}) = t - e^{\psi(t)}, \quad t > 0. \tag{2.2}$$

Differentiating both sides of this equation, we get

$$\theta'(e^{\psi(t)}) = \frac{1}{\psi'(t)\,e^{\psi(t)}} - 1. \tag{2.3}$$

Thus by Lemma 1.2, we have $\theta'(e^{\psi(t)}) > 0$ for all $t > 0$. But since the mapping $t \to e^{\psi(t)}$ from $(0, \infty)$ to $(0, \infty)$ is also bijective this implies that $\theta'(t) > 0$ for all $t > 0$, proving P_1. It is known that, for all $t > 0$

$$\psi(t) < \log(t) - \frac{1}{2t}$$

see [12, (2.11)] and

$$\psi(t) > \log t - \frac{1}{2t} - \frac{1}{12t^2}, \quad t > 0$$

see [7]. By using these two inequalities we obtain that

$$t - te^{-1/\,(2t)} < \theta(e^{\psi(t)}) = t - e^{\psi(t)} < t - te^{-1/\,(2t) - 1/\,(12t^2)}.$$

We can easily check that both of the bounds here tend to $1/2$ as x tends to infinity. Therefore, we have

$$\lim_{u\to\infty} \theta(e^{\psi(u)}) = \lim_{t\to\infty} \theta(t) = \frac{1}{2}.$$

Differentiating both sides of (2.3), we obtain that

$$\theta''(e^{\psi(t)}) = -\frac{e^{-2\psi(t)}}{\psi'(t)^3}[(\psi'(t))^2 + \psi''(t)].$$

By Lemma 1.1 $[\psi'(t)]^2 + \psi''(t) > 0$ for all $t > 0$, hence, we find from this equality that $\theta''(e^{\psi(t)}) < 0$ for all $t > 0$. Proceeding as above we conclude that $\theta''(t) < 0$ for $t > 0$. This proves P_3. P_4 follows immediately from (2.3) and the limit in (1.6).

Let $e^{-\gamma} \leq t < \infty$, then by the monotonicity of θ and property P_2 of θ, we find that

$$1 - e^{-\gamma} = \theta(e^{-\gamma}) \leq \theta(t) < \theta(\infty) = \frac{1}{2}. \tag{2.4}$$

From (2.1) we can write

$$\theta(t) = \psi^{-1}(\log t) - t. \tag{2.5}$$

Substituting the value of $\theta(t)$ into (2.4), we get

$$1 - e^{-\gamma} \leq \psi^{-1}(\log t) - t < 0.5.$$

From the right inequality we get for $x > 0.5$

$$\psi(x) > \log(x - 0.5),$$

and similarly the left inequality gives for $x \geq 1$

$$\psi(x) \leq \log\left(x - 1 + e^{-\gamma}\right).$$

This proves a). In order to prove b) and c) we apply the mean value theorem to θ on the interval $[e^{\psi(t)}, e^{\psi(t+1)}]$. Thus, there exists a δ such that $0 < \delta(t) < 1$ for all $t > 0$ and

$$\theta\left(e^{\psi(t+1)}\right) - \theta\left(e^{\psi(t)}\right) = \left(e^{\psi(t+1)} - e^{\psi(t)}\right)\theta'(e^{\psi(t+\delta(t))}),$$

which can be rewritten by (2.2) as

$$\frac{1}{e^{\psi(t)}(e^{1/t} - 1)} - 1 = \theta'(e^{\psi(t+\delta(t))}). \tag{2.6}$$

By P_1, the right-hand-side of this equation is greater than 0, which proves the right inequality in b) by direct computation. It is clear that

$$\theta\left(e^{\psi(t+1)}\right) - \theta\left(e^{\psi(t)}\right) = 1 - e^{\psi(t)}(e^{1/t} - 1) < \theta(\infty) - \theta(0) = \frac{1}{2}, \quad t > 0.$$

After some simplification this proves the left inequality in b).

Since for $t > 2$, $t + \delta(t) > 1 + \delta(1)$ and θ' is strictly decreasing on $(0, \infty)$ by P_3, we must have for $t > 2$ that

$$\theta'(e^{\psi(t+\delta(t))}) < \theta'(e^{\psi(1)}) = \theta'(e^{-\gamma}) = \frac{6e^{\gamma}}{\pi^2} - 1. \tag{2.7}$$

Making use of (2.6) proves c).

We now prove e). By applying the mean value theorem to θ on the interval $[e^{\psi(t)}, e^{\psi(t+h)}]$ $(t > 0, h > 0)$, we get

$$\theta(e^{\psi(t+h)}) - \theta(e^{\psi(t)}) = (e^{\psi(t+h)} - e^{\psi(t)})\theta'(e^{\psi(t+a)}),$$

where $0 < a < h$. Employing (2.2) and (2.3) , this can be written as

$$\frac{h}{e^{\psi(t+h)} - e^{\psi(t)}} - 1 = \theta'(e^{\psi(t+a)}). \tag{2.8}$$

By the monotonicity of θ and ψ, we have $\theta'(e^{\psi(t+a)}) < \theta'(e^{\psi(t)})$ and $\theta'(e^{\psi(t+a)}) > \theta'(e^{\psi(t+h)})$. Thus by the above inequality and these two inequalities we find that

$$\frac{h}{e^{\psi(t+h)} - e^{\psi(t)}} - 1 < \theta'(e^{\psi(t)}) = \frac{1}{\psi'(t)e^{\psi(t)}} - 1$$

and

$$\frac{h}{e^{\psi(t+h)} - e^{\psi(t)}} - 1 > \theta'(e^{\psi(t+h)}) = \frac{1}{\psi'(t+h)e^{\psi(t+h)}} - 1.$$

After brief computation, these inequalities yield

$$\psi(x+h) - \psi(x) > \log(1 + h\psi'(x)) \tag{2.9}$$

and

$$\psi(x+h) - \psi(x) < -\log(1 - h\psi'(x+h)). \tag{2.10}$$

These prove e).

Put $h = 1$ in (2.9) and (2.10) and then use (1.2) and (1.3) to get after some computations

$$\psi'(x) < e^{1/x} - 1$$

and

$$\psi'(x) > 1 + \frac{1}{x^2} - e^{-1/x}.$$

In the first inequality replace x by $x + 1$ and use (1.2) to get

$$\psi'(x) < \frac{1}{x^2} - 1 + e^{1/(x+1)}.$$

These prove f). By (2.3) we have for $t \geq 1$

$$\theta'(e^{\psi(t)}) = \frac{1}{\psi'(t)e^{\psi(t)}} - 1 < \theta'(e^{\psi(1)}) = \theta'(e^{-\gamma}) = \frac{6e^{\gamma}}{\pi^2} - 1.$$

From these inequalities we obtain after simple computations that, for $t \geq 1$

$$\psi'(t) \geq \frac{\pi^2}{6e^{\gamma}} e^{-\psi(t)}, \tag{2.11}$$

and this proves d).

To prove g) and h) we apply the mean value theorem to $\psi(t + \theta\,(t))$ ($t > 0$) in (2.1) on the interval $[\,0, \theta\,(t)]$ to find that

$$\log t = \psi(t) + \theta(t)\psi'(t + \alpha(t)),$$

where $0 < \alpha(t) < \theta(t)$. Since θ is strictly increasing and ψ' is strictly decreasing on $(0, \infty)$, and $\theta(1) = c - 1$ by (2.5), this gives for $t > 1$

$$\log t - \psi(t) < \frac{1}{2}\psi'(t)$$

and

$$\log t - \psi(t) > (c - 1)\psi'\left(t + \frac{1}{2}\right).$$

From these two equations with the help of f) we prove h) and i).

In order to prove i) and j) integrate both sides of (2.1) over $1 \leq u \leq x$ to obtain

$$\int_1^x \psi(u + \theta(u))du = \int_1^x \log u du.$$

Making the change of variable $u = e^{\psi(t)}$ on the left hand side this becomes by (2.1)

$$\int_c^{x+\theta(x)} \psi(t)\psi'(t)e^{\psi(t)}dt = x \log x - x + 1. \tag{2.12}$$

Since $\psi(t) \geq 0$ for all $t \geq c$, and $\psi'(t)e^{\psi(t)} < 1$ by Lemma 1.2 we find that, for $x > 1$

$$x \log x - x + 1 < \int_c^{x+\theta(x)} \psi(t)dt = \log \Gamma(x + \theta(x)) - \log \Gamma(c)$$

or

$$x \log x - x + 1 + \log \Gamma(c) < \log \Gamma(x + \theta(x)).$$

Again using the monotonicity of θ, this can be rewritten after some simplifications as for $x \geq \frac{1}{2}$

$$\Gamma(x + 1) > \Gamma(c)\left(x + \frac{1}{2}\right)^{x+1/2} e^{-x+1/2}.$$

This proves i). By (2.11) and (2.12) we have for $x \geq 1$ that

$$x\log x - x + 1 \geq \frac{\pi^2}{6e^\gamma}\int_c^{x+\theta(x)} \psi(t)dt = \frac{\pi^2}{6e^\gamma}\log\Gamma(x+\theta(x)) - \frac{\pi^2}{6e^\gamma}\log\Gamma(c)$$

or

$$x\log x - x + 1 + \frac{\pi^2}{6e^\gamma}\log\Gamma(c) \geq \frac{\pi^2}{6e^\gamma}\log\Gamma(x+\theta(x)).$$

Since for $x \geq 1$, $\theta(x) \geq c-1$ from this inequality we find that

$$\frac{6e^\gamma}{\pi^2}x\log x - \frac{6e^\gamma}{\pi^2}x + \frac{6e^\gamma}{\pi^2} + \log\Gamma(c) \geq \log\Gamma(x+c-1).$$

Replacing x by $x-c+2$ we get for $x \geq c-1$

$$\Gamma(x+1) \leq \Gamma(c)(x+2-c)^{6(x+2-c)e^\gamma/\pi^2} e^{6(-x-1+c)e^\gamma/\pi^2},$$

which proves j). Thus, we have completed the proof of the theorem. □

Corollary 2.2. *For any integer $n \geq 1$ the following inequalities involving harmonic numbers and factorial hold.*

a.

$$\gamma + \log(n+0.5) < H_n \leq \gamma + \log(n-1+e^{1-\gamma}).$$

The constants 0.5 and $e^{1-\gamma}-1$ are the best possible.

b.

$$\log\left(\frac{\pi^2}{6}\right) - \log(e^{1/(n+1)}-1) < H_n < \gamma - \log[e^{1/(n+1)}-1].$$

c.

$$n! > \Gamma(c)\left(n+\frac{1}{2}\right)^{n+1/2} e^{-n+1/2}$$

and

$$n! < \Gamma(c)(n+2-c)^{6(n+2-c)e^\gamma/\pi^2} e^{6(-n-1+c)e^\gamma/\pi^2},$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the n^{th} harmonic number.

Proof. Let $x \geq 2$. Then by (2.2) we have

$$\theta(e^{\psi(x)}) = x - e^{\psi(x)} \geq \theta(e^{\psi(2)}) = 2 - e^{\psi(2)} = 2 - e^{1-\gamma}.$$

Thus a short calculation gives for $x \geq 2$

$$\psi(x) \leq \log\left(x-2+e^{1-\gamma}\right).$$

It is well known that $\psi(n+1) = H_n - \gamma$ for all integers $n \geq 1$ (see [1, p. 258, 6.3.2]), thus replacing x by $n+1$ here proves a). Using the identity $\psi(n+1) = H_n - \gamma$ again, the proof of b) follows from Theorem 2.1b by replacing x by $n+1$. c) follows, too, from replacing x by a natural number n since $\Gamma(n+1) = n!$. This completes the proof of the corollary. □

Now define

$$\alpha_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log\left(n+\frac{1}{2}\right)$$

and

$$\beta_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \log(e^{1/(n+1)}-1).$$

Clearly $\lim_{n\to\infty}\alpha_n = \gamma$. Since

$$\lim_{n\to\infty}\left[\log\left(e^{1/(n+1)}-1\right) + \log\left(n+\frac{1}{2}\right)\right] = 0,$$

it is also obvious that $\lim_{n\to\infty}\beta_n = \gamma$.

Thus the arithmetic mean of α_n and β_n converges to γ as well. We define

$$\gamma_n = \frac{\alpha_n + \beta_n}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{2}\log\left(\frac{e^{1/(n+1)} - 1}{n + 1/2}\right).$$

The rate of convergence of α_n has been investigated by De Temple and he has shown that

$$\frac{1}{24(n+1)^2} < \alpha_n - \gamma < \frac{1}{24n^2},$$

see [11]. We have not investigated the rate of convergence of β_n and γ_n, but numerical experiments indicate as illustrated on the following table that β_n converges to γ more rapidly than α_n and, γ_n converges to γ much more rapidly than both α_n and β_n.

Table 2.1: Comparison between some terms of α_n , β_n and γ_n.

n	α_n	β_n	γ_n	$\lvert\alpha_n - \gamma\rvert$	$\lvert\beta_n - \gamma\rvert$	$\lvert\gamma_n - \gamma\rvert$
1	0.594534891	0.567247870	0.580891381	0.017319226	0.009967794	0.003675716
2	0.583709268	0.572679728	0.578194498	0.006493603	0.004535936	0.000978833
3	0.580570364	0.574641783	0.577606074	0.003354699	0.002573881	0.000390409
4	0.579255936	0.575561532	0.577408734	0.002040271	0.001654132	0.000193069
5	0.578585241	0.576064337	0.577324789	0.001369576	0.001151327	0.000109124
10	0.577592996	0.576871855	0.577232426	0.000377331	0.000343809	0.000016761
50	0.577232002	0.577199646	0.577215824	0.000016337	0.000016018	0.000000159
100	0.577219790	0.577211580	0.577215655	0.000004125	0.000004084	0.000000020
500	0.577215831	0.577215498	0.577215685	0.000000166	0.000000166	0.000000000
1000	0.577215706	0.577215623	0.577215666	0.000000041	0.000000041	0.000000000

3. CONCLUSION

We want to make some remarks on our results.

i) Numerical experiments indicate that the function $x \to \theta'(x)$ is strictly completely monotonic, but it seems difficult to prove this. For example, even to prove that $\theta'''(x) > 0$ $(x > 0)$, we need to show the following complicated inequality.

$$\psi'(x)\psi'''(x) - 3\left(\psi''(x)\right)^2 - 3\left(\psi'(x)\right)^2\psi''(x) - 2\left(\psi'(x)\right)^4 < 0, \quad x > 0$$

If we prove this, applying the mean value theorem to $\theta^{(n)}$ for all positive integers n on $[e^{\psi(t)}, e^{\psi(t+1)}]$, we may obtain many other interesting inequalities involving polygamma functions.

ii) In our method presented here we used the mean value theorem. Instead, by using Taylor Theorem up to higher derivatives, we may get sharpenings of the bounds we find here. For example, by applying the Taylor Theorem to $\log\Gamma(x)$ on $[t, t+1]$ $(t > 0)$ up to the second derivative, we get

$$\log t = \psi(t) + \frac{1}{2}\psi'(t + \alpha(t)), \quad 0 < \alpha(t) < 1.$$

Investigating the monotonicity property and the limit of $\alpha(t)$ will be very interesting and can lead to very sharp inequalities for polygamma functions. We showed that the limit of $\alpha(t)$ as t tends to ∞ is $1/3$ provided that this limit exists .

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