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ON SOME NEW FRACTIONAL INTEGRAL INEQUALITIES

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ABSTRACT. In this paper, using the Riemann-Liouville fractional integral, we establish some new integral inequalities for the Chebyshev functional in the case of two synchronous functions.

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1. INTRODUCTION

Let us consider the functional [1]:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

where f and g are two integrable functions which are synchronous on $[a,b]$ $\Big($i.e. $(f(x) - f(y))(g(x)-g(y)) \geq 0$, for any $x, y \in [a,b]\Big)$.

Many researchers have given considerable attention to (1.1) and a number of inequalities have appeared in the literature, see [3, 4, 5].

The main purpose of this paper is to establish some inequalities for the functional (1.1) using fractional integrals.

2. DESCRIPTION OF FRACTIONAL CALCULUS

We will give the necessary notation and basic definitions below. For more details, one can consult [2, 6].

Definition 2.1. A real valued function $f(t)$, $t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.

Definition 2.2. A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, n \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.

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Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau; \qquad \alpha > 0, t > 0, \tag{2.1}$$
$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \qquad \alpha \geq 0, \beta \geq 0, \tag{2.2}$$

which implies the commutative property:

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{2.3}$$

From (2.1), when $f(t) = t^\mu$ we get another expression that will be used later:

$$J^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}, \qquad \alpha > 0; \;\; \mu > -1, t > 0. \tag{2.4}$$

3. MAIN RESULTS

Theorem 3.1. *Let f and g be two synchronous functions on $[0, \infty[$. Then for all $t > 0, \alpha > 0$, we have:*

$$J^\alpha(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} J^\alpha f(t) J^\alpha g(t). \tag{3.1}$$

Proof. Since the functions f and g are synchronous on $[0, \infty[$, then for all $\tau \geq 0, \rho \geq 0$, we have

$$\Big(f(\tau) - f(\rho)\Big)\Big(g(\tau) - g(\rho)\Big) \geq 0. \tag{3.2}$$

Therefore

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.3}$$

Now, multiplying both sides of (3.3) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, we get

$$\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau) + \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho) \geq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\rho) + \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\tau). \tag{3.4}$$

Then integrating (3.4) over $(0,t)$, we obtain:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)g(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\rho)g(\rho)d\tau \geq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)g(\rho)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\rho)g(\tau)d\tau. \tag{3.5}$$

Consequently,

$$J^\alpha(fg)(t) + f(\rho)\, g(\rho) \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \geq \frac{g(\rho)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau + \frac{f(\rho)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau. \tag{3.6}$$

So we have

$$J^{\alpha}(fg)(t) + f(\rho)\, g(\rho)\, J^{\alpha}(1) \geq g(\rho)\, J^{\alpha}(f)(t) + f(\rho)\, J^{\alpha}(g)(t). \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$, $\rho \in (0,t)$, we obtain:

$$\begin{aligned} \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} J^{\alpha}(fg)(t) &+ \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)\, g(\rho)\, J^{\alpha}(1) \\ &\geq \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g(\rho)\, J^{\alpha} f(t) + \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)\, J^{\alpha} g(t). \end{aligned} \tag{3.8}$$

Now integrating (3.8) over $(0,t)$, we get:

$$\begin{aligned} J^{\alpha}(fg)(t) \int_0^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} d\rho &+ \frac{J^{\alpha}(1)}{\Gamma(\alpha)} \int_0^t f(\rho) g(\rho) (t-\rho)^{\alpha-1} d\rho \\ &\geq \frac{J^{\alpha} f(t)}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} g(\rho) d\rho + \frac{J^{\alpha} g(t)}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} f(\rho) d\rho. \end{aligned} \tag{3.9}$$

Hence

$$J^{\alpha}(fg)(t) \geq \frac{1}{J^{\alpha}(1)} J^{\alpha} f(t) J^{\alpha} g(t), \tag{3.10}$$

and this ends the proof. □

The second result is:

Theorem 3.2. *Let f and g be two synchronous functions on $[0,\infty[$. Then for all $t > 0$, $\alpha > 0$, $\beta > 0$, we have:*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(fg)(t) \geq J^{\alpha} f(t) J^{\beta} g(t) + J^{\beta} f(t) J^{\alpha} g(t). \tag{3.11}$$

Proof. Using similar arguments as in the proof of Theorem 3.1, we can write

$$\begin{aligned} \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} J^{\alpha}(fg)(t) &+ J^{\alpha}(1) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho)\, g(\rho) \\ &\geq \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} g(\rho)\, J^{\alpha} f(t) + \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho)\, J^{\alpha} g(t). \end{aligned} \tag{3.12}$$

By integrating (3.12) over $(0,t)$, we obtain

$$\begin{aligned} J^{\alpha}(fg)(t) \int_0^t \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} d\rho &+ \frac{J^{\alpha}(1)}{\Gamma(\beta)} \int_0^t f(\rho)\, g(\rho)\, (t-\rho)^{\beta-1}\, d\rho \\ &\geq \frac{J^{\alpha} f(t)}{\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1}\, g(\rho)\, d\rho + \frac{J^{\alpha} g(t)}{\Gamma(\beta)} \int_0^t (t-\rho)^{\beta-1}\, f(\rho)\, d\rho, \end{aligned} \tag{3.13}$$

and this ends the proof. □

Remark 1. The inequalities (3.1) and (3.11) are reversed if the functions are asynchronous on $[0,\infty[$ (i.e. $(f(x)-f(y))(g(x)-g(y)) \leq 0$, for any $x, y \in [0,\infty[$).

Remark 2. Applying Theorem 3.2 for $\alpha = \beta$, we obtain Theorem 3.1.

The third result is:

Theorem 3.3. *Let $(f_i)_{i=1,\dots,n}$ be n positive increasing functions on $[0,\infty[$. Then for any $t>0, \alpha>0$, we have*

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(J^{\alpha}(1)\right)^{1-n} \prod_{i=1}^{n} J^{\alpha} f_i(t). \tag{3.14}$$

Proof. We prove this theorem by induction.

Clearly, for $n=1$, we have $J^{\alpha}(f_1)(t) \geq J^{\alpha}(f_1)(t)$, for all $t>0, \alpha>0$.

For $n=2$, applying (3.1), we obtain:

$$J^{\alpha}(f_1 f_2)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}(f_1)(t)\, J^{\alpha}(f_2)(t), \qquad \text{for all } t>0, \alpha>0.$$

Now, suppose that (induction hypothesis)

$$J^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \left(J^{\alpha}(1)\right)^{2-n} \prod_{i=1}^{n-1} J^{\alpha} f_i(t), \qquad t>0, \alpha>0. \tag{3.15}$$

Since $(f_i)_{i=1,\dots,n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is an increasing function. Hence we can apply Theorem 3.1 to the functions $\prod_{i=1}^{n-1} f_i = g$, $f_n = f$. We obtain:

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) = J^{\alpha}(fg)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t)\, J^{\alpha}(f_n)(t). \tag{3.16}$$

Taking into account the hypothesis (3.15), we obtain:

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(J^{\alpha}(1)\right)^{-1} \left(\left(J^{\alpha}(1)\right)^{2-n} \left(\prod_{i=1}^{n-1} J^{\alpha} f_i\right)(t)\right) J^{\alpha}(f_n)(t), \tag{3.17}$$

and this ends the proof. □

We further have:

Theorem 3.4. *Let f and g be two functions defined on $[0,+\infty[$, such that f is increasing, g is differentiable and there exists a real number $m := \inf_{t\geq 0} g'(t)$. Then the inequality*

$$J^{\alpha}(fg)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{mt}{\alpha+1} J^{\alpha}f(t) + mJ^{\alpha}(tf(t)) \tag{3.18}$$

is valid for all $t>0, \alpha>0$.

Proof. We consider the function $h(t) := g(t) - mt$. It is clear that h is differentiable and it is increasing on $[0,+\infty[$. Then using Theorem 3.1, we can write:

$$\begin{aligned} J^{\alpha}\Big((g-mt)f(t)\Big) &\geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t)\Big(J^{\alpha}g(t) - mJ^{\alpha}(t)\Big) \\ &\geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{m\left(J^{\alpha}(1)\right)^{-1} t^{\alpha+1}}{\Gamma(\alpha+2)} J^{\alpha}f(t) \\ &\geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{m\Gamma(\alpha+1)\, t}{\Gamma(\alpha+2)} J^{\alpha}f(t) \\ &\geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{mt}{\alpha+1} J^{\alpha}f(t). \end{aligned} \tag{3.19}$$

Hence

$$J^{\alpha}(fg)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{mt}{\alpha+1} J^{\alpha}f(t) + mJ^{\alpha}(tf(t)), \quad t>0, \alpha>0. \tag{3.20}$$

Theorem 3.4 is thus proved. □

Corollary 3.5. *Let f and g be two functions defined on $[0,+\infty[$.*

(A) *Suppose that f is decreasing, g is differentiable and there exists a real number $M := \sup_{t\geq 0} g'(t)$. Then for all $t>0$, $\alpha>0$, we have:*

$$J^{\alpha}(fg)(t) \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) - \frac{Mt}{\alpha+1}J^{\alpha}f(t) + MJ^{\alpha}(tf(t)). \tag{3.21}$$

(B) *Suppose that f and g are differentiable and there exist $m_1 := \inf_{t\geq 0} f'(x)$, $m_2 := \inf_{t\geq 0} g'(t)$. Then we have*

$$\begin{aligned} &J^{\alpha}(fg)(t) - m_1 J^{\alpha}tg(t) - m_2 J^{\alpha}tf(t) + m_1 m_2 J^{\alpha}t^2 \\ &\geq (J^{\alpha}(1))^{-1}\Big(J^{\alpha}f(t)J^{\alpha}g(t) - m_1 J^{\alpha}tJ^{\alpha}g(t) - m_2 J^{\alpha}tJ^{\alpha}f(t) + m_1 m_2 (J^{\alpha}t)^2\Big). \end{aligned} \tag{3.22}$$

(C) *Suppose that f and g are differentiable and there exist $M_1 := \sup_{t\geq 0} f'(t)$, $M_2 := \sup_{t\geq 0} g'(t)$. Then the inequality*

$$\begin{aligned} &J^{\alpha}(fg)(t) - M_1 J^{\alpha}tg(t) - M_2 J^{\alpha}tf(t) + M_1 M_2 J^{\alpha}t^2 \\ &\geq (J^{\alpha}(1))^{-1}\Big(J^{\alpha}f(t)J^{\alpha}g(t) - M_1 J^{\alpha}tJ^{\alpha}g(t) - M_2 J^{\alpha}tJ^{\alpha}f(t) + M_1 M_2 (J^{\alpha}t)^2\Big). \end{aligned} \tag{3.23}$$

is valid.

Proof.
(A): Apply Theorem 3.1 to the functions f and $G(t) := g(t) - m_2 t$.
(B): Apply Theorem 3.1 to the functions F and G, where: $F(t) := f(t) - m_1 t,\ G(t) := g(t) - m_2 t$.
To prove (C), we apply Theorem 3.1 to the functions

$$F(t) := f(t) - M_1 t,\ G(t) := g(t) - M_2 t.$$

□

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