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SOME RESULTS ON THE COMPLEX OSCILLATION THEORY OF DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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ABSTRACT. In this paper, we study the possible orders of transcendental solutions of the differential equation $f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = 0$, where $a_0(z), \ldots, a_{n-1}(z)$ are nonconstant polynomials. We also investigate the possible orders and exponents of convergence of distinct zeros of solutions of non-homogeneous differential equation $f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = b(z)$, where $a_0(z), \ldots, a_{n-1}(z)$ and $b(z)$ are nonconstant polynomials. Several examples are given.

Key words and phrases: Differential equations, Order of growth, Exponent of convergence of distinct zeros, Wiman-Valiron theory.

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1. INTRODUCTION

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [3]). Let $\sigma(f)$ denote the order of an entire function f, that is,

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r}, \tag{1.1}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [3]), and $M(r,f) = \max_{|z|=r} |f(z)|$.

We recall the following definition.

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Definition 1.1. Let f be an entire function. Then the exponent of convergence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}(f) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}. \tag{1.2}$$

We define the logarithmic measure of a set $E \subset [1, +\infty[$ by $lm(E) = \int_1^{+\infty} \frac{\chi_E(t)\, dt}{t}$, where χ_E is the characteristic function of set E.

In the study of the differential equations,

$$f'' + a_1(z) f' + a_0(z) f = 0, \quad f'' + a_1(z) f' + a_0(z) f = b(z), \tag{1.3}$$

where $a_0(z)$, $a_1(z)$ and $b(z)$ are nonconstant polynomials, Z.-X. Chen and C.-C. Yang proved the following results:

Theorem 1.1 ([1]). *Let a_0 and a_1 be nonconstant polynomials with degrees $\deg a_j = n_j$ $(j = 0, 1)$. Let $f(z)$ be an entire solution of the differential equation*

$$f'' + a_1(z) f' + a_0(z) f = 0. \tag{1.4}$$

Then

- (i) *If $n_0 \geq 2n_1$, then any entire solution $f \not\equiv 0$ of the equation (1.4) satisfies $\sigma(f) = \frac{n_0+2}{2}$.*
- (ii) *If $n_0 < n_1 - 1$, then any entire solution $f \not\equiv 0$ of (1.4) satisfies $\sigma(f) = n_1 + 1$.*
- (iii) *If $n_1 - 1 \leq n_0 < 2n_1$, then any entire solution of (1.4) satisfies either $\sigma(f) = n_1 + 1$ or $\sigma(f) = n_0 - n_1 + 1$.*
- (iv) *In (iii), if $n_0 = n_1 - 1$, then the equation (1.4) possibly has polynomial solutions, and any two polynomial solutions of (1.4) are linearly dependent, all the polynomial solutions have the form $f_c(z) = cp(z)$, where p is some polynomial, c is an arbitrary constant.*

Theorem 1.2 ([1]). *Let a_0, a_1 and b be nonconstant polynomials with degrees $\deg a_j = n_j$ $(j = 0, 1)$. Let $f \not\equiv 0$ be an entire solution of the differential equation*

$$f'' + a_1(z) f' + a_0(z) f = b(z). \tag{1.5}$$

Then

- (i) *If $n_0 \geq 2n_1$, then $\overline{\lambda}(f) = \sigma(f) = \frac{n_0+2}{2}$.*
- (ii) *If $n_0 < n_1 - 1$, then $\overline{\lambda}(f) = \sigma(f) = n_1 + 1$.*
- (iii) *If $n_1 - 1 < n_0 < 2n_1$, then $\overline{\lambda}(f) = \sigma(f) = n_1 + 1$ or $\overline{\lambda}(f) = \sigma(f) = n_0 - n_1 + 1$, with at most one exceptional polynomial solution f_0 for three cases above.*
- (iv) *If $n_0 = n_1 - 1$, then every transcendental entire solution f satisfies $\overline{\lambda}(f) = \sigma(f) = n_1 + 1$ (or 0).*

Remark 1.3. If the corresponding homogeneous equation of (1.5) has a polynomial solution $p(z)$, then (1.5) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (1.5), c is a constant). If the corresponding homogeneous equation of (1.5) has no polynomial solution, then (1.5) has at most one polynomial solution.

2. STATEMENT AND PROOF OF RESULTS

For $n \geq 2$, we consider the linear differential equation

$$f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = 0, \tag{2.1}$$

where $a_0(z), \ldots, a_{n-1}(z)$ are nonconstant polynomials with degrees $\deg a_j = d_j$ $(j = 0, \ldots, n-1)$. It is well-known that all solutions of equation (2.1) are entire functions of finite rational order see [7], [6, pp. 106-108], [8, pp. 65-67]. It is also known [5, p. 127], that for any solution f of (2.1), we have

(2.2) $$\sigma(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{d_k}{n-k}.$$

Recently G. Gundersen, M. Steinbart and S. Wang have investigated the possible orders of solutions of equation (2.1) in [2]. In the present paper, we prove two theorems which are analogous to Theorem 1.1 and Theorem 1.2 for higher order linear differential equations.

Theorem 2.1. *Let* $a_0(z), \ldots, a_{n-1}(z)$ *be nonconstant polynomials with degrees* $\deg a_j = d_j$ $(j = 0, 1, \ldots, n-1)$. *Let* $f(z)$ *be an entire solution of the differential equation*

(2.3) $$f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = 0.$$

Then

- (i) *If* $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ *holds for all* $j = 1, \ldots, n-1$*, then any entire solution* $f \not\equiv 0$ *of the equation* (2.3) *satisfies* $\sigma(f) = \frac{d_0+n}{n}$.
- (ii) *If* $d_j < d_{n-1} - (n-j-1)$ *holds for all* $j = 0, \ldots, n-2$*, then any entire solution* $f \not\equiv 0$ *of* (2.3) *satisfies* $\sigma(f) = 1 + d_{n-1}$.
- (iii) *If* $d_j - 1 \leq d_{j-1} < d_j + d_{n-1}$ *holds for all* $j = 1, \ldots, n-1$ *with* $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ *and* $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ *for all* $0 \leq k < j-1$*, then the possible orders of any solution* $f \not\equiv 0$ *of* (2.3) *are:*
$$1 + d_{n-1}, 1 + d_{n-2} - d_{n-1}, \ldots, 1 + d_{j-1} - d_j, \ldots, 1 + d_0 - d_1.$$
- (iv) *In* (iii)*, if* $d_{j-1} = d_j - 1$ *for all* $j = 1, \ldots, n-1$*, then the equation* (2.3) *possibly has polynomial solutions, and any* n *polynomial solutions of* (2.3) *are linearly dependent, all the polynomial solutions have the form* $f_c(z) = cp(z)$*, where* p *is some polynomial,* c *is an arbitrary constant.*

Theorem 2.2. *Let* $a_0(z), \ldots, a_{n-1}(z)$ *and* $b(z)$ *be nonconstant polynomials with degrees* $\deg a_j = d_j$ $(j = 0, 1, \ldots, n-1)$. *Let* $f \not\equiv 0$ *be an entire solution of the differential equation*

(2.4) $$f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = b(z).$$

Then

- (i) *If* $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ *holds for all* $j = 1, \ldots, n-1$*, then* $\overline{\lambda}(f) = \sigma(f) = \frac{d_0+n}{n}$.
- (ii) *If* $d_j < d_{n-1} - (n-j-1)$ *holds for all* $j = 0, \ldots, n-2$*, then* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$.
- (iii) *If* $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ *holds for all* $j = 1, \ldots, n-1$ *with* $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ *and* $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ *for all* $0 \leq k < j-1$*, then* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$ *or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{n-2} - d_{n-1}$ *or ... or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{j-1} - d_j$ *or ... or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_0 - d_1$*, with at most one exceptional polynomial solution* f_0 *for three cases above.*
- (iv) *If* $d_{j-1} = d_j - 1$ *for some* $j = 1, \ldots, n-1$*, then any transcendental entire solution* f *of* (2.4) *satisfies* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$ *or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{n-2} - d_{n-1}$ *or ... or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_j - d_{j+1}$ *or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_{j-2} - d_{j-1}$ *or ... or* $\overline{\lambda}(f) = \sigma(f) = 1 + d_0 - d_1$ *(or* 0*).*

Remark 2.3. If the corresponding homogeneous equation of (2.4) has a polynomial solution $p(z)$, then (2.4) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (2.4), c is a constant). If the corresponding homogeneous equation of (2.4) has no polynomial solution, then (2.4) has at most one polynomial solution.

3. PROOF OF THEOREM 2.1

Assume that $f(z)$ is a transcendental entire solution of (2.3). First of all from the Wiman-Valiron theory (see [4] or [6]), it follows that there exists a set E_1 that has finite logarithmic measure, such that for all $j = 1, \dots, n$ we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \tag{3.1}$$

as $r \to +\infty$, $r \notin E_1$, where $|z| = r$ and $|f(z)| = M(r, f)$. Here $\nu_f(r)$ denotes the central index of f. Furthermore

$$\nu_f(r) = (1 + o(1))\,\alpha r^{\sigma} \tag{3.2}$$

as $r \to +\infty$, where $\sigma = \sigma(f)$ and α is a positive constant. Now we divide equation (2.3) by f, and then substitute (3.1) and (3.2) into (2.3). This yields an equation whose right side is zero and whose left side consists of a sum of $(n+1)$ terms whose absolute values are asymptotic as $(r \to +\infty, r \notin E_1)$ to the following $(n+1)$ terms:

$$\alpha^n r^{n(\sigma-1)}, \beta_{n-1} r^{d_{n-1}+(n-1)(\sigma-1)}, \dots, \beta_j r^{d_j + j(\sigma-1)}, \dots, \beta_0 r^{d_0} \tag{3.3}$$

where $\beta_j = \alpha^j |b_j|$ and $a_j = b_j z^{d_j}(1 + o(1))$ for each $j = 0, \dots, n-1$.

(i) If $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \dots, n-1$, then

$$\sigma(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{d_k}{n-k} = 1 + \frac{d_0}{n}. \tag{3.4}$$

Suppose that $\sigma(f) < 1 + \frac{d_0}{n}$, then we have

$$d_j + j(\sigma - 1) < \left(\frac{n-j}{n}\right) d_0 + j\frac{d_0}{n} = d_0 \tag{3.5}$$

for all $j = 1, \dots, n-1$. Then the term in (3.3) with exponent d_0 is a dominant term as $(r \to +\infty, r \notin E_1)$. This is impossible. Hence $\sigma(f) = 1 + \frac{d_0}{n}$.

(ii) If $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \dots, n-2$, then we have

$$\frac{d_j}{n-j} < \frac{d_{n-1} - (n-j-1)}{n-j} < \frac{d_{n-1}}{n-j} < d_{n-1} \tag{3.6}$$

for all $j = 0, \dots, n-2$. Hence $\max\limits_{0 \leq j \leq n-1} \frac{d_j}{n-j} = d_{n-1}$ and $\sigma(f) \leq 1 + d_{n-1}$. Suppose that $\sigma(f) < 1 + d_{n-1}$. We have for all $j = 0, \dots, n-2$,

$$\begin{aligned} d_j + j(\sigma - 1) &< d_{n-1} - (n-j-1) + j(\sigma - 1) \\ &< d_{n-1} - (n-j-1) + j(\sigma-1) + (n-j-1)\sigma \\ &\leq d_{n-1} + (n-1)(\sigma - 1). \end{aligned} \tag{3.7}$$

Then the term in (3.3) with exponent $d_{n-1} + (n-1)(\sigma - 1)$ is a dominant term as $(r \to +\infty, r \notin E_1)$. This is impossible. Hence $\sigma(f) = 1 + d_{n-1}$.

(iii) If $d_j - 1 \leq d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \ldots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, then we have in this case

$$\max_{0 \leq j \leq n-1} \frac{d_j}{n-j} = d_{n-1}. \tag{3.8}$$

Hence $\sigma(f) \leq 1 + d_{n-1}$. Set

$$\sigma_j = 1 + d_{j-1} - d_j \qquad (j = 1, \ldots, n-1) \tag{3.9}$$

and

$$\sigma_n = 1 + d_{n-1}. \tag{3.10}$$

First, we prove that $\sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n$. From the conditions, we have

$$d_{j-1} - d_j > \frac{d_{j-2} - d_j}{2} \qquad (j = 2, \ldots, n-1), \tag{3.11}$$

which yields

$$-(j-2)\, d_{j-1} - d_j > d_{j-2} - j d_{j-1}. \tag{3.12}$$

Adding $(j-1)\, d_{j-1}$ to both sides of (3.12) gives

$$d_{j-1} - d_j > d_{j-2} - d_{j-1} \qquad (j = 2, \ldots, n-1). \tag{3.13}$$

Hence $\sigma_{j-1} < \sigma_j$ for all $j = 2, \ldots, n-1$. Furthermore, from the conditions, we have $d_{j-1} - d_j < d_{n-1}$ for all $j = 1, \ldots, n-1$. Hence $\sigma_j < \sigma_n$ for all $j = 1, \ldots, n-1$. Finally, we obtain that $\sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n$. Next suppose $\sigma_j < \sigma < \sigma_{j+1}$ for some $j = 1, \ldots, n-1$.

(a) First we prove that if $\sigma > \sigma_j$ for some $j = 1, \ldots, n-1$, and k is any integer satisfying $0 \leq k < j$, then $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$. Since

$$d_k + k(\sigma - 1) = d_j + j(\sigma - 1) + d_k - d_j + (k-j)(\sigma - 1), \tag{3.14}$$

we obtain

$$d_k + k(\sigma - 1) < d_j + j(\sigma - 1) + d_k - d_j + (k-j)(\sigma_j - 1). \tag{3.15}$$

Now from the definition of σ_j in (3.9), we obtain

$$d_k - d_j + (k-j)(\sigma_j - 1) = (k-j)\left[d_{j-1} - d_j - \frac{d_k - d_j}{j-k}\right]. \tag{3.16}$$

Since $0 \leq k < j$, it follows from the conditions that

$$d_{j-1} - d_j \geq \frac{d_k - d_j}{j-k}. \tag{3.17}$$

Then from (3.16) and (3.17), we obtain that

$$d_k - d_j + (k-j)(\sigma_j - 1) \leq 0. \tag{3.18}$$

Hence $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for all $0 \leq k < j$.

(b) Now, we prove that if $\sigma < \sigma_{j+1}$ for some $j = 0, \ldots, n-1$ and k is any integer satisfying $j < k \leq n-1$, then $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$. First, remark that if $k = j+1$, then

$$\begin{aligned} d_{j+1} + (j+1)(\sigma - 1) &= d_{j+1} + (\sigma - 1) + j(\sigma - 1) \\ &< d_{j+1} + (\sigma_{j+1} - 1) + j(\sigma - 1) \\ &= d_{j+1} + (d_j - d_{j+1}) + j(\sigma - 1) \\ &\leq d_j + j(\sigma - 1). \end{aligned}$$

Hence

$$d_{j+1} + (j+1)(\sigma - 1) < d_j + j(\sigma - 1). \tag{3.19}$$

We have,

$$\sigma < \sigma_{j+1} < \sigma_{j+2} < \cdots < \sigma_{n-1} < \sigma_n. \tag{3.20}$$

Then

$$\begin{aligned} d_{j+2} + (j+2)(\sigma - 1) &< d_{j+1} + (j+1)(\sigma - 1)\ (\sigma < \sigma_{j+2}) \\ &\cdots \\ d_{n-1} + (n-1)(\sigma - 1) &< d_{n-2} + (n-2)(\sigma - 1)\ (\sigma < \sigma_{n-1}). \end{aligned} \tag{3.21}$$

Therefore from (3.20) and by combining the inequalities in (3.19) and (3.21), we obtain that $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for all $j < k \leq n-1$. Furthermore

$$n(\sigma - 1) = (n-1)(\sigma - 1) + (\sigma - 1) < (n-1)(\sigma - 1) + d_{n-1}$$

since $\sigma < \sigma_n$ and from (3.21) and (3.19), we deduce that $n(\sigma - 1) < d_j + j(\sigma - 1)$. Then from $a)$ and $b)$, we obtain that if $\sigma_j < \sigma < \sigma_{j+1}$ for some $j = 1, \ldots, n-1$, then $n(\sigma - 1) < d_j + j(\sigma - 1)$ and $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for any $k \neq j$. It follows that the term in (3.3) with exponent $d_j + j(\sigma - 1)$ is a dominant term (as $r \to +\infty, r \notin E_1$). This is impossible. From $b)$, it follows that if $\sigma < \sigma_1$, then $d_k + k(\sigma - 1) < d_0$ for all $0 < k \leq n-1$ and $n(\sigma - 1) < d_0$. Hence the term in (3.3) with exponent d_0 is a dominant term (as $r \to +\infty, r \notin E_1$). This is impossible.
Finally, we deduce that the possible orders of f are

$$1 + d_{n-1}, 1 + d_{n-2} - d_{n-1}, \ldots, 1 + d_{j-1} - d_j, \ldots, 1 + d_0 - d_1.$$

(iv) If $d_{j-1} = d_j - 1$ for all $j = 1, \ldots, n-1$, it is easy to see that (2.3) has possibly polynomial solutions. Now we discuss polynomial solutions of equation (2.3), if $f_1(z), \ldots, f_n(z)$ are linearly independent polynomial solutions, then by the well-known identity

$$\begin{vmatrix} f_1 & f_2 & f_n \\ f_1' & f_2' & f_n' \\ & \cdots & \\ f_1^{(n-1)} & f_2^{(n-1)} & f_n^{(n-1)} \end{vmatrix} = C \exp\left\{ -\int_0^z a_{n-1}(s)\, ds \right\}, \tag{3.22}$$

where $C \neq 0$ is some constant, we obtain a contradiction. Therefore any n polynomial solutions are linearly dependent, hence all polynomial solutions have the form $f_c(z) = cp(z)$, where p is a polynomial and c is an arbitrary constant.

Next, we give several examples that illustrate the sharpness of Theorem 2.1.

Example 3.1. Consider the differential equation

$$f''' - (6z+1)\, f'' + 3z\,(3z+1)\, f' - 2\left(z^3 + z^2 - 1\right) f = 0. \tag{3.23}$$

Set

$$\begin{aligned} a_2(z) &= -(6z+1), \qquad d_2 = 1; \\ a_1(z) &= 3z\,(3z+1), \qquad d_1 = 2; \\ a_0(z) &= -2\left(z^3 + z^2 - 1\right), \qquad d_0 = 3. \end{aligned}$$

We have $\frac{d_0}{3} \geq \frac{d_1}{2}$ and $\frac{d_0}{3} \geq \frac{d_2}{1}$. Hence, by Theorem 2.1$(i)$, all transcendental solutions of equation (3.23) are of order $1 + \frac{d_0}{3} = 2$. We see for example that $f(z) = e^{z^2}$ is a solution of (3.23) with $\sigma(f) = 2$.

Example 3.2. Consider the differential equation

$$f''' + zf'' + 2\left(z^2 - 8z - 1\right) f' - 3\left(9z^6 + 3z^5 + 2z^4 + 2z^3 + 2\right) f = 0. \tag{3.24}$$

Set

$$\begin{aligned} a_2(z) &= z, \qquad d_2 = 1; \\ a_1(z) &= 2\left(z^2 - 8z - 1\right), \qquad d_1 = 2; \\ a_0(z) &= -3\left(9z^6 + 3z^5 + 2z^4 + 2z^3 + 2\right), \qquad d_0 = 6. \end{aligned}$$

We have $\frac{d_0}{3} > \frac{d_1}{2}$ and $\frac{d_0}{3} > \frac{d_2}{1}$. Hence, by Theorem 2.1$(i)$, all transcendental solutions of equation (3.24) are of order $1 + \frac{d_0}{3} = 3$. Remark that $f(z) = e^{z^3}$ is a solution of (3.24) with $\sigma(f) = 3$.

Example 3.3. Consider the differential equation

$$f'''' - 2zf''' - 4\left(z^2 + 1\right) f'' + 6z^3 f' + 4\left(z^4 - 1\right) f = 0. \tag{3.25}$$

Set

$$\begin{aligned} a_3(z) &= -2z, \qquad d_3 = 1; \\ a_2(z) &= -4\left(z^2 + 1\right), \qquad d_2 = 2; \\ a_1(z) &= 6z^3, \qquad d_1 = 3; \\ a_0(z) &= 4\left(z^4 - 1\right), \qquad d_0 = 4. \end{aligned}$$

We have $\frac{d_j}{4-j} \leq \frac{d_0}{4}$ for all $j = 1, 2, 3$. Hence, by Theorem 2.1(i), all transcendental solutions of equation (3.25) are of order $1 + \frac{d_0}{4} = 2$. Remark that $f(z) = e^{z^2}$ is a solution of (3.25) with $\sigma(f) = 2$.

Example 3.4. Consider the differential equation

$$f''' + \left(z^2 + z - 1\right) f'' + \left(z^3 - z^2 - z + 1\right) f' - \left(z^3 + 1\right) f = 0. \tag{3.26}$$

Set

$$\begin{aligned} a_2(z) &= z^2 + z - 1, \qquad d_2 = 2; \\ a_1(z) &= z^3 - z^2 - z + 1, \qquad d_1 = 3; \\ a_0(z) &= -\left(z^3 + 1\right), \qquad d_0 = 3. \end{aligned}$$

We have $d_1 - 1 < d_0 < d_1 + d_2$ and $d_2 - 1 < d_1 < 2d_2$, $d_1 - d_2 > \frac{d_0 - d_2}{2}$. Hence, by Theorem 2.1$(iii)$, all possible orders of solutions of equation (3.26) are $1 + d_2 = 3$, $1 + d_1 - d_2 = 2$, $1 + d_0 - d_1 = 1$. For example $f(z) = e^z$ is a solution of (3.26) with $\sigma(f) = 1$.

Example 3.5. The equation

$$f''' + z^3 f'' - 2z^2 f' + 2zf = 0$$

has a polynomial solution $f_c(z) = c(z^2 + 2z)$ where c is a constant.

Example 3.6. The equation

$$f'''' - z\left(z^3 + 3z^2 + 2z + 1\right) f''' - z\left(z^2 + 3z + 1\right) f'' + 2\left(z^2 + z + 1\right) f' + 6(z + 1) f = 0$$

has a polynomial solution $f_c(z) = c(z^3 + 3z^2)$ where c is a constant.

4. PROOF OF THEOREM 2.2

We assume that $f(z)$ is a transcendental entire solution of (2.4).We adopt the argument as used in the proof of Theorem 2.1, and notice that when z satisfies $|f(z)| = M(r, f)$ and $|z| \to +\infty$, $\left|\frac{b(z)}{f(z)}\right| \to 0$, we can prove that

(1) if $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \ldots, n-1$, then $\sigma(f) = \frac{d_0+n}{n}$;
(2) if $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \ldots, n-2$, then $\sigma(f) = 1 + d_{n-1}$;
(3) if $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \ldots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, then $\sigma(f) = 1 + d_{n-1}$ or $\sigma(f) = 1 + d_{n-2} - d_{n-1}$ or ... or $\sigma(f) = 1 + d_{j-1} - d_j$ or ... or $\sigma(f) = 1 + d_1 - d_2$ or $\sigma(f) = 1 + d_0 - d_1$.

We know that when $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \ldots, n-1$ or $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \ldots, n-2$ or $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \ldots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, every solution $f \not\equiv 0$ of the corresponding homogeneous equation of (2.4) is transcendental, so that the equation (2.4) has at most one exceptional polynomial solution, in fact if f_1, f_2 $(f_2 \not\equiv f_1)$ are polynomial solutions of (2.4), then $f_1 - f_2 \not\equiv 0$ is a polynomial solution of the corresponding homogeneous equation of (2.4), this is a contradiction. When $d_{j-1} = d_j - 1$ for some $j = 1, \ldots, n-1$, if the corresponding homogeneous equation of (2.4) has no polynomial solution, then (2.4) has clearly at most one exceptional polynomial solution, if the corresponding homogeneous equation of (2.4) has a polynomial solution $p(z)$, then (2.4) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (2.4), c is a constant). Now we prove $\overline{\lambda}(f) = \sigma(f)$ for a transcendental solution f of (2.4). Since $b(z)$ is a polynomial which has only finitely many zeros, it follows that if z_0 is a zero of $f(z)$ and $|z_0|$ is sufficiently large, then the order of zero at z_0 is less than or equal to n from (2.4). Hence

$$N\left(r, \frac{1}{f}\right) \leq n\overline{N}\left(r, \frac{1}{f}\right) + O(\ln r). \tag{4.1}$$

By (2.4), we have

$$\frac{1}{f} = \frac{1}{b}\left(\frac{f^{(n)}}{f} + a_{n-1}\frac{f^{(n-1)}}{f} + \cdots + a_1\frac{f'}{f} + a_0\right). \tag{4.2}$$

Hence

$$m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(\ln r). \tag{4.3}$$

By $\sigma(f) < +\infty$, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(\ln r)\,(j = 1, \dots, n). \tag{4.4}$$

Then we get from (4.1), (4.3) and (4.4),

$$\begin{aligned} T(r,f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq n\,\overline{N}\left(r, \frac{1}{f}\right) + d(\log r) \end{aligned} \tag{4.5}$$

where $d\,(> 0)$ is a constant. By (4.5), we have $\sigma(f) \leq \overline{\lambda}(f)$. On the other hand, we have

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) \tag{4.6}$$

since $m\left(r, \frac{1}{f}\right)$ is a positive function. Hence

$$\overline{N}\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{1}{f}\right) = T(r,f) + O(1). \tag{4.7}$$

From (4.7), we obtain $\overline{\lambda}(f) \leq \sigma(f)$. Therefore, $\overline{\lambda}(f) = \sigma(f)$.

Next, we give several examples that illustrate the sharpness of Theorem 2.2.

Example 4.1. Consider the differential equation

$$f''' - (6z+1) f'' + 3z(3z+1) f' - 2\left(z^3 + z^2 - 1\right) f = z\left(-2z^3 - 2z^2 + 9z + 5\right). \tag{4.8}$$

By Theorem 2.2(i), every entire transcendental solution of equation (4.8) is of order $1 + \frac{d_0}{3} = 2$. Remark that $f(z) = z + e^{z^2}$ is a solution of (4.8) with $\sigma(f) = \overline{\lambda}(f) = 2$.

Example 4.2. Consider the differential equation

$$f'''' - 2zf''' - 4\left(z^2 + 1\right) f'' + 6z^3 f' + 4\left(z^4 - 1\right) f = 4\left(z^6 + 3z^4 - 3z^2 - 2\right). \tag{4.9}$$

From Theorem 2.2(i), it follows that every entire transcendental solution of equation (4.9) is of order $1 + \frac{d_0}{4} = 2$. We have $f(z) = z^2 + e^{z^2}$ is a solution of (4.9) with $\sigma(f) = \overline{\lambda}(f) = 2$.

Example 4.3. Consider the differential equation

$$f''' + \left(z^2 + z - 1\right) f'' + \left(z^3 - z^2 - z + 1\right) f' - \left(z^3 + 1\right) f = z^4 - z^3 + z^2 + 2z - 1. \tag{4.10}$$

If f is a solution of equation (4.10), then by Theorem 2.2(iii), it follows that $\sigma(f) = \overline{\lambda}(f) = 3$ or $\sigma(f) = \overline{\lambda}(f) = 2$ or $\sigma(f) = \overline{\lambda}(f) = 1$. We have for example $f(z) = -z + e^z$ is a solution of (4.10) with $\sigma(f) = \overline{\lambda}(f) = 1$.

Example 4.4. The equation

$$f''' + \left(z^3 + z^2 + z + 1\right) f'' - \left(2z^2 + 2z + 1\right) f' + 2(z+1) f = 2(z+1)$$

has a family of polynomial solutions $\{c\,(z^2 + 2z) + 1\}$ (c is a constant).

Example 4.5. The equation

$$f''' + \left(z^3 + z^2 + z + 1\right) f'' - \left(2z^2 + 2z + 1\right) f' + 2(z+1) f = 4z + 3$$

has a family of polynomial solutions $\{c\,(z^2 + 2z) + z + 2\}$ (c is a constant).

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