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SOME NEW HARDY TYPE INEQUALITIES AND THEIR LIMITING INEQUALITIES

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ABSTRACT. A new necessary and sufficient condition for the weighted Hardy inequality is proved for the case $1 < p \leq q < \infty$. The corresponding limiting Pólya-Knopp inequality is also proved for $0 < p \leq q < \infty$. Moreover, a corresponding limiting result in two dimensions is proved. This result may be regarded as an endpoint inequality of Sawyer's two-dimensional Hardy inequality. But here we need only one condition to characterize the inequality whereas in Sawyer's case three conditions are necessary.

Key words and phrases: Inequalities, Weights, Hardy's inequality, Pólya-Knopp's inequality, Multidimensional inequalities, Sawyer's two-dimensional operator.

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1. INTRODUCTION

We are inspired by the clever Hardy-Pólya observation to the Hardy inequality,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,\ f \geq 0,\ p > 1,$$

that by changing f to $f^{\frac{1}{p}}$ and tending $p \to \infty$ we obtain the Pólya-Knopp inequality

$$\int_0^\infty Gf(x)dx \leq e\int_0^\infty f(x)dx$$

with the geometric mean operator

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right).$$

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Let $1 < p \le q < \infty$. In particular, in [3] the authors tried to find a new condition for the weighted inequality

$$\left(\int_0^\infty \left(\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1.1}$$

by using the weighted Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1.2}$$

and replacing $u(x)$ and $f(x)$ by $u(x)x^{-q}$ and $f(x)$ by $f^\alpha(x)$ respectively in (1.2). Then, by replacing q with $\frac{q}{\alpha}$ and p with $\frac{p}{\alpha}$ so that (1.2) becomes

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{q}{\alpha}} u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}$$

and letting $\alpha \to 0$ we obtain (1.1). The natural choice was of course to try to use the usual "Muckenhoupt" condition (see [4] and [6])

$$A_M = \sup_{x>0}\left(\int_x^\infty u(t)dt\right)^{\frac{1}{q}}\left(\int_0^x v(t)^{1-p'}dt\right)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1},$$

which, with the same substitutions, will be

$$A_M(\alpha) = \sup_{x>0}\left(\int_x^\infty u(t)t^{-\frac{q}{\alpha}}dt\right)^{\frac{1}{q}}\left(\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{p-\alpha}{\alpha p}} < \infty.$$

However, as $\alpha \to 0$ the first term tends to 0. By making a suitable change in the condition $A_M(\alpha)$ the author was able to give a sufficient condition. In [7] this problem was solved in a satisfactory way by first proving a new necessary and sufficient condition for the weighted Hardy inequality (1.2), namely

$$A_{P.S} = \sup_{x>0} V(x)^{-\frac{1}{p}}\left(\int_0^x u(t)V(t)^q dt\right)^{\frac{1}{q}} < \infty,$$

where $V(x) = \int_0^x v(t)^{1-p'}dt$, which was also already proved in the case $p = q$ in [9], and the bounds for the best possible constant C in (1.2) are

$$A_{P.S} \le C \le p' A_{P.S}.$$

Then, by using the limiting procedure described above, they obtained the following necessary and sufficient condition for the inequality (1.1) to hold for $0 < p \le q < \infty$:

$$D_{P.S} = \sup_{x>0} x^{-\frac{1}{p}}\left(\int_0^x w(t)dt\right)^{\frac{1}{q}} < \infty,$$

where

$$w(t) = \left(\exp\left(\frac{1}{t}\int_0^t \ln\frac{1}{v(y)}dy\right)\right)^{\frac{q}{p}} u(t) \tag{1.3}$$

and

$$D_{P.S} \le C \le e^{\frac{1}{p}} D_{P.S}.$$

The lower bound was also proved in a lemma ([7, Lemma 1]) directly by using the following test function:

$$f(x) = t^{-\frac{1}{p}}\chi_{[0,t]}(x) + (xe)^{-\frac{s}{p}}t^{\frac{s-1}{p}}\chi_{(t,\infty)}(x), \quad s > 1, t > 0.$$

By omitting different intervals the following two lower bounds were pointed out:

$$D_{P.S} \leq C$$

and

$$\sup_{s>1}\left(\frac{(s-1)e^s)}{1+(s-1)e^s}\right)^{\frac{1}{p}}\sup_{t>0} t^{\frac{s-1}{p}}\left(\int_t^\infty \frac{w(x)}{x^{\frac{sq}{p}}}dx\right)^{\frac{1}{q}} \leq C.$$

Moreover, in [5] the following theorem was proved:

Theorem 1.1. *Let $0 < p \leq q < \infty$ and u, v be weight functions. Then there exists a positive constant $C < \infty$ such that the inequality (1.1) holds for all $f > 0$ if and only if there is a $s > 1$ such that*

$$D_{O.G} = D_{O.G}(s,q,p) = \sup_{t>0} t^{\frac{s-1}{p}}\left(\int_t^\infty \frac{w(x)}{x^{\frac{sq}{p}}}dx\right)^{\frac{1}{q}} < \infty, \tag{1.4}$$

where $w(x)$ is defined by (1.3). Moreover, if C is the least constant for which (1.1) holds, then

$$\sup_{s>1}\left(\frac{s-1}{s}\right)^{\frac{1}{p}} D_{O.G}(s,q,p) \leq C \leq \inf_{s>1} e^{\frac{s-1}{p}} D_{O.G}(s,q,p).$$

We see that the condition (1.4) and the condition from Lemma 1 in [7] is the same but the lower bound is different. Since

$$\left(\frac{(s-1)e^s)}{1+(s-1)e^s}\right)^{\frac{1}{p}} - \left(\frac{s-1}{s}\right)^{\frac{1}{p}} = \left(\frac{s-1}{s}\right)^{\frac{1}{p}}\left(\left(\frac{se^s}{se^s+(1-e^s)}\right)^{\frac{1}{p}} - 1\right) > 0 \tag{1.5}$$

for all $s > 1$, we note that the lower bound from Lemma 1 in [7] is better than that from [5]. This suggests that the bounds for the best constant C in (1.1) with the condition (1.4) should be

$$\sup_{s>1}\left(\frac{(s-1)e^s)}{1+(s-1)e^s}\right)^{\frac{1}{p}} D_{O.G}(s,q,p) \leq C \inf_{s>1} e^{\frac{s-1}{p}} D_{O.G}(s,q,p). \tag{1.6}$$

In [5] the authors also proved that for the upper bound s should be $s = 1 + \frac{p}{q}$. Thus, the best possible bounds for the constant C should be

$$\sup_{s>1}\left(\frac{(s-1)e^s)}{1+(s-1)e^s}\right)^{\frac{1}{p}} D_{O.G}(s,q,p) \leq C \leq e^{\frac{1}{q}} D_{O.G}(1+\frac{p}{q},q,p). \tag{1.7}$$

In Section 2 of this paper we prove a new necessary and sufficient condition for the weighted Hardy inequality (1.2) to hold (see Theorem 1). In Section 3 we make the limiting procedure described above in our new Hardy inequality and obtain condition (1.4) for the inequality (1.1) to hold and we also receive the expected bounds as in (1.6) or (1.7) (see Theorem 2). Finally, in Section 4 we prove that a two-dimensional version of the inequality (1.1) can be characterized by a two-dimensional version of the condition (1.4) and, moreover, the bounds corresponding to (1.6) or (1.7) hold (see Theorem 3). This result fits perfectly as an end point inequality of the Hardy inequalities proved by E. Sawyer ([8], Theorem 1) for the (rectangular) Hardy operator

$$H(f)(x_1x_2) = \int_0^x\int_0^x f(t_1,t_2)dt_1dt_2.$$

We note that for the Hardy case E. Sawyer showed that three conditions were necessary to characterize the inequality but in our endpoint case only one condition is necessary and sufficient. In Section 5 we give some concluding remarks, shortly discuss the different weight condition for characterizing the Hardy inequality and prove a two-dimensional Minkowski inequality we needed for the proof of Theorem 3 but which is also of independent interest (see Proposition 1).

2. A New Weight Characterization of Hardy's Inequality

Our main theorem in this section reads:

Theorem 2.1. *Let* $1 < p \leq q < \infty$, *and* $s \in (1, p)$ *then the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{2.1}$$

holds for all $f \geq 0$ *iff*

$$A_W(s,q,p) = \sup_{t>0} V(t)^{\frac{s-1}{p}} \left(\int_t^\infty u(x)V(x)^{q\left(\frac{p-s}{p}\right)}dx\right)^{\frac{1}{q}} < \infty, \tag{2.2}$$

where $V(t) = \int_0^t v(x)^{1-p'}dx$. *Moreover if* C *is the best possible constant in (2.1) then*

$$\sup_{1<s<p} \left(\frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}}\right)^{\frac{1}{p}} A_W(s,q,p) \leq C \leq \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A_W(s,q,p). \tag{2.3}$$

Proof of Theorem 2.1. Let $f^p(x)v(x) = g(x)$ in (2.1). Then (2.1) is equivalent to

$$\left(\int_0^\infty \left(\int_0^x g(t)^{\frac{1}{p}} v(t)^{-\frac{1}{p}}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty g(x)dx\right)^{\frac{1}{p}}. \tag{2.4}$$

Assume that (2.2) holds. We have, by applying Hölder's inequality, the fact that $DV(t) = v(t)^{1-p'} = v(t)^{-\frac{p'}{p}}$ and Minkowski's inequality,

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^x g(t)^{\frac{1}{p}} v(t)^{-\frac{1}{p}}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\
&\quad = \left(\int_0^\infty \left(\int_0^x g(t)^{\frac{1}{p}} V(t)^{\frac{s-1}{p}} V(t)^{-\frac{s-1}{p}} v(t)^{-\frac{1}{p}}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\
&\quad \leq \left(\int_0^\infty \left(\int_0^x g(t)V(t)^{s-1}dt\right)^{\frac{q}{p}} \left(\int_0^x V(t)^{-\frac{(s-1)p'}{p}} v(t)^{-\frac{p'}{p}}dt\right)^{\frac{q}{p'}} u(x)dx\right)^{\frac{1}{q}} \\
&\quad = \left(\frac{p}{p-(s-1)p'}\right)^{\frac{1}{p'}} \left(\int_0^\infty \left(\int_0^x g(t)V(t)^{s-1}dt\right)^{\frac{q}{p}} V(x)^{\frac{p-(s-1)p'}{p}\frac{q}{p'}} u(x)dx\right)^{\frac{1}{q}} \\
&\quad \leq \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} \left(\int_0^\infty g(t)V(t)^{s-1} \left(\int_t^\infty V(x)^{q\left(\frac{p-s}{p}\right)} u(x)dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \\
&\quad \leq \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A_W(s,q,p) \left(\int_0^\infty g(t)dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence (2.4) and thus (2.1) holds with a constant satisfying the right hand side inequality in (2.3).

Now we assume that (2.1) and thus (2.4) holds and choose the test function

$$g(x) = \left(\frac{p}{p-s}\right)^p V(t)^{-s} v(x)^{1-p'} \chi_{(0,t)}(x) + V(x)^{-s} v(x)^{1-p'} \chi_{(t,\infty)}(x),$$

where t is a fixed number > 0. Then the right hand side is equal to

$$\left(\int_0^t \left(\frac{p}{p-s}\right)^p V(t)^{-s} v(x)^{1-p'} dx + \int_t^\infty V(x)^{-s} v(x)^{1-p'} dx\right)^{\frac{1}{p}} \\ \leq \left(\left(\frac{p}{p-s}\right)^p V(t)^{1-s} - \frac{1}{1-s} V(t)^{1-s}\right)^{\frac{1}{p}}$$

Moreover, the left hand side is greater than

$$\left(\int_t^\infty \left(\int_0^t \frac{p}{p-s} V(t)^{-\frac{s}{p}} v(y)^{1-p'} dy + \int_t^x V(y)^{-\frac{s}{p}} v(y)^{1-p'} dy\right)^q u(x) dx\right)^{\frac{1}{q}} \\ = \left(\int_t^\infty \left(\frac{p}{p-s}\right)^q V(x)^{\left(1-\frac{s}{p}\right)q} u(x) dx\right)^{\frac{1}{q}}.$$

Hence, (2.4) implies that

$$\left(\frac{p}{p-s}\right)\left(\int_t^\infty V(x)^{\left(1-\frac{s}{p}\right)q} u(x) dx\right)^{\frac{1}{q}} \leq C\left(\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}\right)^{\frac{1}{p}} V(t)^{\frac{1-s}{p}}$$

i.e., that

$$\left(\frac{p}{p-s}\right)\left(\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}\right)^{-\frac{1}{p}} V(t)^{\frac{s-1}{p}} \left(\int_t^\infty V(x)^{\left(1-\frac{s}{p}\right)q} u(x) dx\right)^{\frac{1}{q}} \leq C$$

or, equivalently, that

$$\left(\frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}}\right)^{\frac{1}{p}} V(t)^{\frac{s-1}{p}} \left(\int_t^\infty V(x)^{\left(1-\frac{s}{p}\right)q} u(x) dx\right)^{\frac{1}{q}} \leq C.$$

We conclude that (2.2) and the left hand side of the estimate of (2.3) hold. The proof is complete. □

Remark 2.2. If we replace the interval $(0,\infty)$ in (2.1) with the interval (a,b), then, by modifying the proof above, we see that Theorem 2.1 is still valid with the same bounds and the condition

$$A_W(s,q,p) = \sup_{t>0} V(t)^{\frac{s-1}{p}} \left(\int_t^b u(x) V(x)^{q\left(\frac{p-s}{p}\right)} dx\right)^{\frac{1}{q}} < \infty,$$

where $V(t) = \int_a^t v(x)^{1-p'} dx$.

3. A WEIGHT CHARACTERIZATION OF PÓLYA-KNOPP'S INEQUALITY

In this section we prove that a slightly improved version of Theorem 1.1 can be obtained just as a natural limit of our Theorem 2.1.

Theorem 3.1. *Let* $0 < p \leq q < \infty$ *and* $s > 1$. *Then the inequality*

$$\left(\int_0^\infty \left(\exp\left(\frac{1}{x}\int_0^x \ln f(t)\right) dt\right)^q u(x) dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x) v(x) dx\right)^{\frac{1}{p}} \tag{3.1}$$

holds for all $f > 0$ if and only if $D_{O.G}(s,q,p) < \infty$, where $D_{O.G}(s,q,p)$ is defined by (1.4). Moreover, if C is the best possible constant in (3.1), then

$$\sup_{s>1}\left(\frac{(s-1)e^s}{1+(s-1)e^s}\right)^{\frac{1}{p}} D_{O.G}(s,q,p) \le C \le e^{\frac{1}{q}} D_{O.G}\left(1+\frac{p}{q},q,p\right). \tag{3.2}$$

Remark 3.2. Theorem 3.1 is due to B. Opic and P. Gurka, but our lower bound in (3.2) is strictly better (see (1.5)). As mentioned before, other weight characterizations of (3.1) have been proved by L.E. Persson and V. Stepanov [7] and H.P. Heinig, R. Kerman and M. Krbec [1].

Proof. If we in the inequality (3.1) replace $f^p(x)v(x)$ with $f^p(x)$ and let $w(x)$ be defined as in (1.3), then we see that (3.1) is equivalent to

$$\left(\int_0^\infty \left(\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}.$$

Further, by using Theorem 2.1 with $u(x) = w(x)x^{-q}$ and $v(x) = 1$, we have that

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \tag{3.3}$$

holds for all $f \ge 0$ if and only if

$$A_W(s,q,p) = \sup_{t>0} t^{\frac{s-1}{p}}\left(\int_t^\infty \frac{w(x)}{x^{\frac{sq}{p}}}dx\right)^{\frac{1}{q}} = D_{O.G}(s,q,p) < \infty. \tag{3.4}$$

Moreover, if C is the best possible constant in (3.3), then

$$\begin{aligned} \sup_{1<s<p}\left(\frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}}\right)^{\frac{1}{p}} D_{O.G}(s,q,p) &\le C \\ &\le \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} D_{O.G}(s,q,p). \end{aligned} \tag{3.5}$$

Now, we replace f in (3.3) with f^α, $0 < \alpha < p$, and after that we replace p with $\frac{p}{\alpha}$ and q with $\frac{q}{\alpha}$ in (3.3) – (3.5), we find that for $1 < s < \frac{p}{\alpha}$

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{q}{\alpha}} w(x)dx\right)^{\frac{1}{q}} \le C_\alpha\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \tag{3.6}$$

holds for all $f \ge 0$ if and only if

$$D_{O.G}\left(s,\frac{q}{\alpha},\frac{p}{\alpha}\right) = D^\alpha_{O.G}(s,q,p) < \infty.$$

Moreover, if C_α is the best possible constant in (3.6), then

$$\sup_{1<s<\frac{p}{\alpha}}\left(\frac{\left(\frac{p}{p-\alpha s}\right)^{\frac{p}{\alpha}}}{\left(\frac{p}{p-\alpha s}\right)^{\frac{p}{\alpha}}+\frac{1}{s-1}}\right)^{\frac{1}{p}} D_{O.G}(s,q,p)\le C \le \inf_{1<s<\frac{p}{\alpha}}\left(\frac{p-\alpha}{p-\alpha s}\right)^{\frac{p-\alpha}{\alpha p}} D_{O.G}(s,q,p). \tag{3.7}$$

We also note that

$$\left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{1}{\alpha}} \downarrow \exp\frac{1}{x}\int_0^x \ln f(t)dt, \text{ as } \alpha\to 0_+.$$

We conclude that (3.1) holds exactly when $\lim_{\alpha\to 0_+} C_\alpha<\infty$ and this holds, according to (3.7), exactly when (3.4) holds. Moreover, when $\alpha\to 0_+$ (3.7) implies that (3.2) holds, where we have inserted the optimal value $s=1+\frac{p}{q}$ on the right hand side as pointed out in [5]. The proof is complete. □

4. WEIGHTED TWO-DIMENSIONAL EXPONENTIAL INEQUALITIES

In [8], E. Sawyer proved a two-dimensional weighted Hardy inequality for the case $1<p\le q<\infty$. More exactly, he showed that for the inequality

$$\left(\int_0^\infty\int_0^\infty\left(\int_0^x\int_0^{x_2} f(t_1,t_2)\,dt_1dt_2\right)^q w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \le C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)\,v(x_2,x_2)\,dx_1dx_2\right)^{\frac{1}{p}}$$

to hold, three different weight conditions must be satisfied. Here we will show that when we consider the endpoint inequality of this Hardy inequality, we only need one weight condition to characterize the inequality. Our main result in this section reads:

Theorem 4.1. *Let* $0<p\le q<\infty$, *and let* u,v *and* f *be positive functions on* $\mathbb{R}^2_+$. *If* $0<b_1,b_2\le\infty$, *then*

$$\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(y_1,y_2)dy_1dy_2\right)\right]^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \le C\left(\int_0^{b_1}\int_0^{b_2} f^p(x_1,x_2)v(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \tag{4.1}$$

if and only if

$$D_W(s_1,s_2,p,q):=\sup_{\substack{y_1\in(0,b_1)\\ y_2\in(0,b_2)}} y_1^{\frac{s_1-1}{p}}y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{-\frac{s_1q}{p}}x_2^{-\frac{s_2q}{p}}w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}<\infty, \tag{4.2}$$

where $s_1,s_2>1$ *and*

$$w(x_1,x_2)=\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_1dt_2\right)\right]^{\frac{q}{p}} u(x_1,x_2)$$

and the best possible constant C in (4.1) can be estimated in the following way:

$$\begin{aligned}(4.3)\qquad \sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2,p,q)\\ \le C\\ \le \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}} D_W(s_1,s_2,p,q).\end{aligned}$$

Remark 4.2. For the case $p=q=1$, $b_1=b_2=\infty$ a similar result was recently proved by H. P. Heinig, R. Kerman and M. Krbec [1] but without the estimates of the operator norm (= the best constant C in (4.1)) pointed out in (4.3) here.

We will need a two-dimensional version of the following well-known Minkowski integral inequality:

$$(4.4)\qquad \left(\int_a^b \Phi(x)\left(\int_a^x \Psi(y)dy\right)^r dx\right)^{\frac{1}{r}} \le \int_a^b \Psi(y)\left(\int_y^b \Phi(x)dx\right)^{\frac{1}{r}} dy.$$

The following proposition will be required in the proof of Theorem 4.1. Proposition 4.3 will be proved in Section 5.

Proposition 4.3. *Let* $r>1$, $a_1,a_2,b_1,b_2\in\mathbb{R}$, $a_1<b_{1,}a_2<b_2$ *and let* Φ *and* Ψ *be measurable functions on* $[a_1,b_1]\times[a_2,b_2]$ *. Then*

$$\begin{aligned}(4.5)\qquad \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Phi(x_1,x_2)\left(\int_{a_1}^{x_1}\int_{a_2}^{x_2}\Psi(y_1,y_2)dy_1dy_2\right)^r dx_1dx_2\right)^{\frac{1}{r}}\\ \le \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Psi(y_1,y_2)\left(\int_{y_1}^{b_1}\int_{y_2}^{b_2}\Phi(x_1,x_2)dx_1d_2\right)^{\frac{1}{r}} dy_1dy_2.\end{aligned}$$

Proof of Theorem 4.1. Assume that (4.2) holds. Let $g(x_1,x_2)=f^p(x_1,x_2)v(x_1,x_2)$ in (4.1):

$$\begin{aligned}\Bigg(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g(y_1,y_2)dy_1dy_2\right)\right]^{\frac{q}{p}}\\ \times\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_1dt_2\right)\right]^{\frac{q}{p}} u(x_1,x_2)dx_1dx_2\Bigg)^{\frac{1}{q}}\\ \le C\left(\int_0^{\infty}\int_0^{\infty} g(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}}.\end{aligned}$$

If we let

$$w(x_1,x_2)=\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_1dt_2\right)\right]^{\frac{q}{p}} u(x_1,x_2),$$

then we can equivalently write (4.1) as

$$\begin{aligned}(4.6)\qquad \left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g(y_1,y_2)dy_1dy_2\right)\right]^{\frac{q}{p}} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\\ \le C\left(\int_0^{b_1}\int_0^{b_2} g(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}}.\end{aligned}$$

Let $y_1 = x_1 t_1$ and $y_2 = x_2 t_2$, then (4.6) becomes

$$\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\int_0^1\int_0^1 \log g(x_1t_1, x_2t_2)dt_1dt_2\right)\right]^{\frac{q}{p}} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \leq C\left(\int_0^{b_1}\int_0^{b_2} g(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}}. \tag{4.7}$$

By using the result

$$\left(\exp\int_0^1\int_0^1 \log t_1^{s_1-1}t_2^{s_2-1}dt_1dt_2\right)^{\frac{q}{p}} = e^{-(s_1+s_2-2)\frac{q}{p}}$$

and Jensen's inequality, the left hand side of (4.7) becomes

$$\begin{aligned}
&e^{\frac{s_1+s_2-2}{p}}\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\int_0^1\int_0^1 \log\left[t_1^{s_1-1}t_2^{s_2-1}g(x_1t_1,x_2t_2)\right]dt_1dt_2\right]^{\frac{q}{p}} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\\
&\leq e^{\frac{s_1+s_2-2}{p}}\left(\int_0^{b_1}\int_0^{b_2}\left[\int_0^1\int_0^1 t_1^{s_1-1}t_2^{s_2-1}g(x_1t_1,x_2t_2)dt_1dt_2\right]^{\frac{q}{p}} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\\
&= e^{\frac{s_1+s_2-2}{p}}\left(\int_0^{b_1}\int_0^{b_2}\left[\int_0^{x_1}\int_0^{x_2} y_1^{s_1-1}y_2^{s_2-1}g(y_1,y_2)dy_1dy_2\right]^{\frac{q}{p}} x_1^{-s_1\frac{q}{p}}x_2^{-s_2\frac{q}{p}}w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore, by also using Minkowski's integral inequality (4.5) for $p < q$ and Fubini's theorem for $p = q$, we find that the left hand side in (4.6) can be estimated as follows:

$$\begin{aligned}
&\leq e^{\frac{s_1+s_2-2}{p}}\left(\int_0^{b_1}\int_0^{b_2} y_1^{s_1-1}y_2^{s_2-1}g(y_1,y_2)\left(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{-s_1\frac{q}{p}}x_2^{-s_2\frac{q}{p}}w(x_1,x_2)dx_1dx_2\right)^{\frac{p}{q}} dy_1dy_2\right)^{\frac{1}{p}}\\
&\leq e^{\frac{s_1+s_2-2}{p}} D_W(s_1,s_2,q,p)\cdot\left(\int_0^{b_1}\int_0^{b_2} g(y_1,y_2)dy_1dy_2\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, (4.6) and, thus, (4.1) holds with a constant C satisfying the right hand side estimate in (4.3).

Now, assume that (4.1) holds. For fixed t_1 and t_2, $0 < t_1 < b_1$, $0 < t_2 < b_2$, we choose the test function

$$\begin{aligned}
g(x_1,x_2) = g_0(x_1,x_2) &= t_1^{-1}t_2^{-1}\chi_{(0,t_1)}(x_1)\,\chi_{(0,t_2)}(x_2)\\
&\quad + t_1^{-1}\chi_{(0,t_1)}(x_1)\,\frac{e^{-s_2}t_2^{s_2-1}}{x_2^{s_2}}\chi_{(t_2,\infty)}(x_2)\\
&\quad + \frac{e^{-s_1}t_1^{s_1-1}}{x_1^{s_1}}\chi_{(t_1,\infty)}(x_1)\,t_2^{-1}\chi_{(0,t_2)}(x_2)\\
&\quad + \frac{e^{-(s_1+s_2)}t_1^{s_1-1}t_2^{s_2-1}}{x_1^{s_1}x_2^{s_2}}\chi_{(t_1,\infty)}(x_1)\,\chi_{(t_2,\infty)}(x_2).
\end{aligned}$$

Then the right side of (4.6) yields

$$\left(\int_0^{b_1}\int_0^{b_2} g_0\left(y_1,y_2\right)dy_1dy_2\right)^{\frac{1}{p}}$$
$$=\left(\int_0^{t_1}\int_0^{t_2} t_1^{-1}t_2^{-1}dy_1dy_2+\int_0^{t_1}\int_{t_2}^{b_2} t_1^{-1}\frac{e^{-s_2}t_2^{s_2-1}}{y_2^{s_2}}dy_1dy_2\right.$$
$$\left.+\int_{t_1}^{b_1}\int_0^{t_2} t_2^{-1}\frac{e^{-s_1}t_1^{s_1-1}}{y_1^{s_1}}dy_1dy_2+\int_{t_1}^{b_1}\int_{t_2}^{b_2}\frac{e^{-(s_1+s_2)}t_1^{s_1-1}t_2^{s_2-1}}{y_1^{s_1}y_2^{s_2}}dy_1dy_2\right)^{\frac{1}{p}}$$
$$=\left(1+\frac{e^{-s_2}}{s_2-1}\left(1-\left(\frac{t_2}{b_2}\right)^{s_2-1}\right)+\frac{e^{-s_1}}{s_1-1}\left(1-\left(\frac{t_1}{b_1}\right)^{s_1-1}\right)\right.$$
$$\left.+\frac{e^{-s_1}e^{-s_2}}{\left(s_1-1\right)\left(s_2-1\right)}\left(1-\left(\frac{t_1}{b_1}\right)^{s_1-1}\right)\left(1-\left(\frac{t_2}{b_2}\right)^{s_2-1}\right)\right)^{\frac{1}{p}}$$
$$\leq\left(1+\frac{e^{-s_2}}{s_2-1}+\frac{e^{-s_1}}{s_1-1}+\frac{e^{-s_1}e^{-s_2}}{\left(s_1-1\right)\left(s_2-1\right)}\right)^{\frac{1}{p}},$$

i.e.,

$$\left(\int_0^{b_1}\int_0^{b_2} g_0\left(y_1,y_2\right)dy_1dy_2\right)^{\frac{1}{p}}\leq\left(\frac{e^{s_1}\left(s_1-1\right)+1}{e^{s_1}\left(s_1-1\right)}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}\left(s_2-1\right)+1}{e^{s_2}\left(s_2-1\right)}\right)^{\frac{1}{p}}. \tag{4.8}$$

Moreover, for the left hand side in (4.6) we have

$$\left(\int_0^{b_1}\int_0^{b_2} w(x_1,x_2)\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g(y_1,y_2)dy_1dy_2\right)\right]^{\frac{q}{p}}dx_1dx_2\right)^{\frac{1}{q}} \tag{4.9}$$
$$\geq\left(\int_{t_1}^{b_1}\int_{t_2}^{b_2} w(x_1,x_2)\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g(y_1,y_2)dy_1dy_2\right)\right]^{\frac{q}{p}}dx_1dx_2\right)^{\frac{1}{q}}.$$

With the function $g_0(y_1,y_2)$ we get that

$$\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g_0(y_1,y_2)dy_1dy_2\right)=\exp\left(I_1+I_2+I_3+I_4\right),$$

where

$$I_1=\frac{1}{x_1x_2}\int_0^{t_1}\int_0^{t_2}\log\left(t_1^{-1}t_2^{-1}\right)dy_1dy_2=-\frac{t_1t_2}{x_1x_2}\log t_1-\frac{t_1t_2}{x_1x_2}\log t_2,$$

$$I_2=\frac{1}{x_1x_2}\int_0^{t_1}\int_{t_2}^{x_2}\log\left(t_1^{-1}\frac{e^{-s_2}t_2^{s_2-1}}{y_2^{s_2}}\right)dy_1dy_2$$
$$=-\frac{t_1}{x_1}\log t_1+\frac{t_1t_2}{x_1x_2}\log t_1+\left(s_2-1\right)\frac{t_1}{x_1}\log t_2+\frac{t_1t_2}{x_1x_2}\log t_2-s_2\frac{t_1}{x_1}\log x_2,$$

$$I_3=\frac{1}{x_1x_2}\int_{t_1}^{x_1}\int_0^{t_2}\log\left(t_2^{-1}\frac{e^{-s_1}t_1^{s_1-1}}{y_1^{s_1}}\right)dy_1dy_2$$
$$=-\frac{t_2}{x_2}\log t_2+\frac{t_1t_2}{x_1x_2}\log t_2+(s_1-1)\frac{t_2}{x_2}\log t_1+\frac{t_1t_2}{x_1x_2}\log t_1-s_1\frac{t_2}{x_2}\log x_1,$$

and

$$\begin{aligned}I_4 &= \frac{1}{x_1x_2}\int_{t_1}^{x_1}\int_{t_2}^{x_2}\log\left(\frac{e^{-(s_1+s_2)}t_1^{s_1-1}t_2^{s_2-1}}{y_1^{s_1}y_2^{s_2}}\right)dy_1dy_2\\ &= (s_1-1)\log t_1 - (s_1-1)\frac{t_2}{x_2}\log t_1 - \frac{t_1t_2}{x_1x_2}\log t_1\\ &\qquad + (s_2-1)\log t_2 - (s_2-1)\frac{t_1}{x_1}\log t_2 - \frac{t_1t_2}{x_1x_2}\log t_2\\ &\qquad - s_1\log x_1 + \frac{t_1}{x_1}\log t_1 + s_1\frac{t_2}{x_2}\log x_1\\ &\qquad - s_2\log x_2 + \frac{t_2}{x_2}\log t_2 + s_2\frac{t_1}{x_1}\log x_2.\end{aligned}$$

Now we see that

$$I_1+I_2+I_3+I_4 = \log\left(\frac{t_1^{(s_1-1)}t_2^{(s_2-1)}}{x_1^{s_1}x_2^{s_2}}\right)$$

so that, by (4.9),

$$\begin{aligned}&\left(\int_0^{b_1}\int_0^{b_2} w(x_1,x_2)\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g_0(y_1,y_2)dy_1dy_2\right)\right]^{\frac{q}{p}}dx_1dx_2\right)^{\frac{1}{q}}\\ &\qquad\qquad \geq \left(\int_{t_1}^{b_1}\int_{t_2}^{b_2} w(x_1,x_2)\left[\frac{t_1^{(s_1-1)}t_2^{(s_2-1)}}{x_1^{s_1}x_2^{s_2}}\right]^{\frac{q}{p}}dx_1dx_2\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, by (4.6) and (4.8),

$$\begin{aligned}&t_1^{\frac{s_1-1}{p}}t_2^{\frac{s_2-1}{p}}\left(\int_{t_1}^{b_1}\int_{t_2}^{b_2} x_1^{-\frac{q}{p}s_1}x_2^{-\frac{q}{p}s_2}w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\\ &\qquad\qquad \leq C\left(\frac{e^{s_1}(s_1-1)+1}{e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)+1}{e^{s_2}(s_2-1)}\right)^{\frac{1}{p}}\end{aligned}$$

i.e.

$$\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2,q,p) \leq C.$$

We conclude that (4.2) holds and that the left hand inequality in (4.3) holds. The proof is complete. □

Corollary 4.4. *Let $0 < p \leq q < \infty$, and let f be a positive function on $\mathbb{R}_+^2$. Then*

$$\begin{aligned}(4.10)\quad &\left(\int_0^{\infty}\int_0^{\infty}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(y_1,y_2)dy_1dy_2\right)\right]^q x_1^{\alpha_1}x_2^{\alpha_2})dx_1dx_2\right)^{\frac{1}{q}}\\ &\qquad\qquad \leq C\left(\int_0^{\infty}\int_0^{\infty} f^p(x_1,x_2)x_1^{\beta_1}x_2^{\beta_2}dx_1dx_2\right)^{\frac{1}{p}}\end{aligned}$$

holds with a finite constant C if and only if

$$\frac{\alpha_1+1}{q} = \frac{\beta_1+1}{p}$$

and

$$\frac{\alpha_2+1}{q}=\frac{\beta_2+1}{p}$$

and the best constant C has the following estimate:

$$\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\cdot\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s_1-1}\cdot\frac{1}{s_2-1}\right)^{\frac{1}{q}}e^{\frac{\beta_1+\beta_2}{p}}\left(\frac{p}{q}\right)^{\frac{2}{q}}$$
$$\le C\le\inf_{s_1,s_2>1}e^{\frac{s_1+s_2}{p}+\frac{2}{q}}.$$

Proof. Apply Theorem 3.1 with the weights $u(x_1,x_2)=x_1^{\alpha_1}x_2^{\alpha_2}$ and $v(x_1,x_2)=x_1^{\beta_1}x_2^{\beta_2}$. □

Remark 4.5. If $p=q$, then the inequality (4.10) is sharp with the constant $C=e^{\frac{\beta_1+\beta_2+2}{p}}$, see Theorem 2.2 in [2]

5. FINAL REMARKS AND PROOF

5.1. **On Minkowski's integral inequalities.** In order to prove the two-dimensional Minkowski integral inequality in Proposition 4.3 we in fact need the following forms of Minkowski's integral inequality in one dimension:

Lemma 5.1.

a) *Let $r>1$, $a,b\in\mathbb{R}$, $a<b$ and $c<d$. If Φ and Ψ are positive measurable functions on $[a,b]$, then (4.4) holds.*

b) *If $K(x,y)$ is a measurable function on $[a,b]\times[c,d]$, then*

$$\left(\int_a^b\left(\int_c^d K(x,y)dy\right)^r dx\right)^{\frac{1}{r}}\le\int_c^d\left(\int_a^b K^r(x,y)dx\right)^{\frac{1}{r}}dy. \tag{5.1}$$

For the reader's convenience we include here a simple proof.

Proof.

b) Let $r'=\frac{r}{r-1}$. By using the sharpness in Hölder's inequality, Fubini's theorem and an obvious estimate we have

$$\begin{aligned}\left(\int_a^b\left(\int_c^d K(x,y)dy\right)^r dx\right)^{\frac{1}{r}} &= \sup_{\|\varphi(x)\|_{L_{r'}(a,b)}\le 1}\int_a^b\varphi(x)\left(\int_c^d K(x,y)dy\right)dx\\ &= \sup_{\|\varphi(x)\|_{L_{r'}(a,b)}\le 1}\int_c^d\left(\int_a^b K(x,y)\varphi(x)dx\right)dy\\ &\le \int_c^d\sup_{\|\varphi(x)\|_{L_{r'}(a,b)}\le 1}\left(\int_a^b K(x,y)\varphi(x)dx\right)dy\\ &= \int_c^d\left(\int_a^b K^r(x,y)dx\right)^{\frac{1}{r}}dy.\end{aligned}$$

a) The proof follows by applying (5.1) with $c=a$, $d=b$ and

$$K(x,y)=\begin{cases}\Phi^{\frac{1}{r}}(x)\Psi(y), & a\le y\le x,\\ 0, & x<y\le b.\end{cases}$$

□

Proof of Proposition 4.3. Put $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$; by the symbol $\mathbf{x} \le \mathbf{y}$ we mean $x_1 \le y_1$ and $x_2 \le y_2$, etc.; $d\mu(\mathbf{y}) = \Psi(\mathbf{y})d\mathbf{y}$ and $d\nu(\mathbf{x}) = \Phi(\mathbf{x})d\mathbf{x}$. Then the inequality (4.5) reads:

$$\left(\int\limits_{\mathbf{a}\le\mathbf{x}\le\mathbf{b}} \left(\int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{x}} d\mu(\mathbf{y}) \right)^r d\nu(\mathbf{x}) \right)^{\frac{1}{r}} \le \int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{b}} \left(\int\limits_{\mathbf{y}\le\mathbf{x}\le\mathbf{b}} d\nu(\mathbf{x}) \right)^{\frac{1}{r}} d\mu(\mathbf{y}).$$

We use the Hölder inequality and the Fubini theorem and get

$$\begin{aligned} \left(\int\limits_{\mathbf{a}\le\mathbf{x}\le\mathbf{b}} \left(\int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{x}} d\mu(\mathbf{y}) \right)^r d\nu(\mathbf{x}) \right)^{\frac{1}{r}} &= \sup_{\|g\|_{L^{r'}(d\nu)}\le 1} \int\limits_{\mathbf{a}\le\mathbf{x}\le\mathbf{b}} g(\mathbf{x}) \left(\int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{x}} d\mu(\mathbf{y}) \right) d\nu(\mathbf{x}) \\ &= \sup_{\|g\|_{L^{r'}(d\nu)}\le 1} \int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{b}} \left(\int\limits_{\mathbf{y}\le\mathbf{x}\le\mathbf{b}} g(\mathbf{x}) d\nu(\mathbf{x}) \right) d\mu(\mathbf{y}) \\ &\le \int\limits_{\mathbf{a}\le\mathbf{y}\le\mathbf{b}} \left(\int\limits_{\mathbf{y}\le\mathbf{x}\le\mathbf{b}} d\nu(\mathbf{x}) \right)^{\frac{1}{r}} d\mu(\mathbf{y}). \end{aligned}$$

The proof is complete. □

Remark 5.2. By using the technique in the proof of Proposition 4.3, we find that the following n-dimensional version of (4.5) holds:

Let $n \in \mathbb{Z}_+$ and $r > 1$. Then

$$\begin{aligned} \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \Phi(x_1, x_2, \dots, x_n) \right. & \\ \times \left(\int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \Psi(y_1, y_2, \dots, y_n) dy_1 \dots dy_n \right)^r & \left. dx_1 \dots dx_n \right)^{\frac{1}{r}} \\ \le \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \Psi(y_1, \dots, y_n) & \\ \times \left(\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \Phi(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \right)^{\frac{1}{r}} & dy_1 \dots dy_n. \end{aligned} \tag{5.2}$$

Remark 5.3. In view of our proof of Theorem 4.1 and Remark 5.2, we find that there exists also a n-dimensional variant of Theorem 4.1 $(n \in \mathbb{Z}_+)$, where the actual endpoint Hardy type inequality can be characterized by only one weight condition.

Remark 5.4. For $r = 1$ the inequalities in (4.4), (4.5), (5.1), and (5.2) are reduced to equalities according to the Fubini theorem.

5.2. **On the conditions and the best constant in the Hardy inequality (2.1):** As we have seen, there are at least three different conditions to characterize the Hardy inequality (2.1) for $1 < p \leq q < \infty$, namely the classical (Muckenhoupt) condition (see [4], [6]), the condition by L.E. Persson and V. Stepanov (see [7]) and the new condition derived in Theorem 2.1. It is difficult to make a comparison in the general case and here we only consider the power weight case, i.e. when $u(x) = x^{a-q}$, $v(x) = x^b$. In this case the inequality (2.1) holds for all $f \geq 0$ iff

$$\frac{a+1}{q} = \frac{b+1}{p}$$

and

$$A_M = \left(\frac{p}{q}\right)^{\frac{1}{q}} \left(\frac{1}{p-(b+1)}\right)^{\frac{1}{q}} \left(\frac{p-1}{p-(b+1)}\right)^{\frac{1}{p'}},$$

$$\begin{aligned} A_{P.S} &= \left(\frac{p}{q}\right)^{\frac{1}{q}} \left(\frac{1}{p-(b+1)}\right)^{\frac{1}{q}} \left(\frac{p-1}{p-(b+1)}\right)^{\frac{1}{p'}} (p-1)^{\frac{1}{q}} \\ &= A_M\,(p-1)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} A_W(s,q,p) &= \left(\frac{p}{q}\right)^{\frac{1}{q}} \left(\frac{1}{p-(b+1)}\right)^{\frac{1}{q}} \left(\frac{p-1}{p-(b+1)}\right)^{\frac{1}{p'}} \left(\frac{p-1}{s-1}\right)^{\frac{1}{q}} \\ &= A_M \left(\frac{p-1}{s-1}\right)^{\frac{1}{q}}. \end{aligned}$$

Moreover, if C is the best constant in (2.1), then C has the following estimates:

$$A_M \leq C \leq \left(1+\frac{q}{p'}\right)^{\frac{1}{q}} \left(1+\frac{p'}{q}\right)^{\frac{1}{p'}} A_M,$$

$$A_{P.S} \leq C \leq p' A_{P.S},$$

and

$$\sup_{1<s<p} \left(\frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}}\right)^{\frac{1}{p}} A_W(s,q,p) \leq C \leq \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A_W(s,q,p).$$

Note in this case, with $s = \frac{p^2+qp-q}{p+qp-q}$, we have from the upper bound, that

$$\begin{aligned} C &\leq \inf_{1<s<p} A_W(s,q,p) \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} \\ &= A_M \inf_{1<s<p} \left(\frac{p-1}{s-1}\right)^{\frac{1}{q}} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} \\ &= \left(1+\frac{q}{p'}\right)^{\frac{1}{q}} \left(1+\frac{p'}{q}\right)^{\frac{1}{p'}} A_M. \end{aligned}$$

We finish this paper by giving a numerical example.

Example 5.1. Let $p = 3, q = 4$, and $s = 1.15$ for the lower bound of $A_W(s,q,p)$. Then with the condition A_M we have the following bounds:

$$A_M \leq C \leq A_M \cdot 1.711077405,$$

with the condition $A_{P.S}$ we have the following bounds:

$$A_M \cdot 1.189207115 \leq C \leq A_M \cdot 1.783810673,$$

and with the condition $A_W(s)$ we have the following bounds:

$$A_M \cdot 1.396254480 \leq C \leq A_M \cdot 1.711077405.$$

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