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SOME INEQUALITIES FOR THE GAMMA FUNCTION

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ABSTRACT. In this paper are established some inequalities involving the Euler gamma function. We use the ideas and methods that were used by J. Sándor in his paper [2].

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1. INTRODUCTION

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

The Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

C. Alsina and M.S. Tomás in [1] proved the following double inequality:

Theorem 1.1. *For all $x \in [0, 1]$ and all nonnegative integers n, the following double inequality is true:*

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1. \tag{1.1}$$

Using the series representation of $\psi(x)$, J. Sándor in [2] proved the following generalized result of (1.1):

Theorem 1.2. *For all $a \ge 1$ and all $x \in [0, 1]$, one has:*

$$\frac{1}{\Gamma(1+a)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1. \tag{1.2}$$

143-07

In this paper, using the series representation of $\psi(x)$ and ideas used in [2] we will establish some double inequalities involving the gamma function, "similar" to (1.2).

2. MAIN RESULTS

In order to establish the proof of the theorems, we need the following lemmas:

Lemma 2.1. *If $x > 0$, then the digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ has the following series representation*

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)}, \tag{2.1}$$

where γ is the Euler's constant.

Proof. See [3]. □

Lemma 2.2. *Let $x \in [0,1]$ and a, b be two positive real numbers such that $a \geq b$. Then*

$$\psi(a+bx) \geq \psi(b+ax). \tag{2.2}$$

Proof. It is easy to verify that $a + bx > 0$, $b + ax > 0$. Then by (2.1) we obtain:

$$\begin{aligned}\psi(a+bx) - \psi(b+ax) &= (a+bx-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(a+bx+k)} \\ &\qquad - (b+ax-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(b+ax+k)} \\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{a+bx-1}{a+bx+k} - \frac{b+ax-1}{b+ax+k}\right) \\ &= \sum_{k=0}^{\infty}\frac{(a-b)(1-x)}{(a+bx+k)(b+ax+k)} \geq 0.\end{aligned}$$

□

Alternative proof of Lemma 2.2. Let $x > 0, y > 0$ and $x \geq y$. Then

$$\begin{aligned}\psi(x) - \psi(y) &= (x-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(x+k)} - (y-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(y+k)} \\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{x-1}{x+k} - \frac{y-1}{y+k}\right) \\ &= \sum_{k=0}^{\infty}\frac{(x-y)}{(x+k)(y+k)} \geq 0.\end{aligned}$$

So $\psi(x) \geq \psi(y)$.

In our case: since $a + bx > 0, b + ax > 0$ it is easy to verify that for $x \in [0,1], a \geq b > 0$ we have $a + bx \geq b + ax$, so $\psi(a + bx) \geq \psi(b + ax)$. □

Lemma 2.3. *Let $x \in [0,1]$, a, b $(a \geq b)$ be two positive real numbers such that $\psi(b + ax) > 0$. Let c, d be two given positive real numbers such that $bc \geq ad > 0$. Then*

$$bc\psi(a+bx) - ad\psi(b+ax) \geq 0. \tag{2.3}$$

Proof. Since $\psi(b+ax)>0$, by (2.2) it is clear that $\psi(a+bx)>0$. Now, since $bc\geq ad$, using Lemma 2.2, we have:

$$bc\psi(a+bx)\geq ad\psi(a+bx)\geq ad\psi(b+ax).$$

So $bc\psi(a+bx)-ad\psi(b+ax)\geq 0$. □

Theorem 2.4. *Let f be a function defined by*

$$f(x)=\frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x\in[0,1]$, $a\geq b>0$, c,d are positive real numbers such that: $bc\geq ad>0$ and $\psi(b+ax)>0$. Then f is an increasing function on $[0,1]$, and the following double inequality holds:

$$\frac{\Gamma(a)^c}{\Gamma(b)^d}\leq\frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d}\leq\frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}.$$

Proof. Let $g(x)$ be a function defined by $g(x)=\log f(x)$. Then:

$$g(x)=c\log\Gamma(a+bx)-d\log\Gamma(b+ax).$$

So

$$g^{'}(x)=bc\frac{\Gamma^{'}(a+bx)}{\Gamma(a+bx)}-ad\frac{\Gamma^{'}(b+ax)}{\Gamma(b+ax)}=bc\psi(a+bx)-ad\psi(b+ax).$$

Using (2.3), we have $g^{'}(x)\geq 0$. It means that $g(x)$ is increasing on $[0,1]$. This implies that $f(x)$ is increasing on $[0,1]$.

So for $x\in[0,1]$ we have $f(0)\leq f(x)\leq f(1)$ or

$$\frac{\Gamma(a)^c}{\Gamma(b)^d}\leq\frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d}\leq\frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}.$$

This concludes the proof of Theorem 2.4. □

In a similar way, it is easy to prove the following lemmas and theorems.

Lemma 2.5. *Let $x\geq 1$ and a,b be two positive real numbers such that $b\geq a$. Then*

$$\psi(a+bx)\geq\psi(b+ax).$$

Lemma 2.6. *Let $x\geq 1,a,b$ $(b\geq a)$ be two positive real numbers such that $\psi(b+ax)>0$ and c,d be any two given real numbers such that $bc\geq ad>0$. Then*

$$bc\psi(a+bx)-ad\psi(b+ax)\geq 0.$$

Theorem 2.7. *Let f be a function defined by*

$$f(x)=\frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x\geq 1,b\geq a>0,c,d$ are positive real numbers such that $bc\geq ad>0$ and $\psi(b+ax)>0$. Then f is an increasing function on $[1,+\infty)$.

Lemma 2.8. *Let $x\in[0,1]$, a,b $(a\geq b)$ be two positive real numbers such that $\psi(a+bx)<0$ and c,d be any two given real numbers such that $ad\geq bc>0$. Then*

$$bc\psi(a+bx)-ad\psi(b+ax)\geq 0.$$

Using Lemmas 2.2 and 2.8, and the methods we used in Theorem 2.4, the following theorem can be proved:

Theorem 2.9. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x \in [0,1]$, $a \geq b > 0$, c,d are positive real numbers such that $ad \geq bc > 0$ and $\psi(a+bx) < 0$. Then f is an increasing function on $[0,1]$.

Lemma 2.10. *Let $x \geq 1, a, b$ $(b \geq a)$ be two positive real numbers such that $\psi(a+bx) < 0$ and c,d be any two given real numbers such that $ad \geq bc > 0$. Then*

$$bc\psi(a+bx) - ad\psi(b+ax) \geq 0.$$

Using Lemmas 2.5 and 2.10, and the methods we used in Theorem 2.4, the following theorem can be proved:

Theorem 2.11. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x > 1, b \geq a > 0$, c,d are positive real numbers such that $ad \geq bc > 0$ and $\psi(a+bx) < 0$. Then f is an increasing function on $[1,+\infty)$.

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