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L_p INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

NISAR A. RATHER

P.G. DEPARTMENT OF MATHEMATICS
KASHMIR UNIVERSITY
HAZRATBAL, SRINAGAR-190006, INDIA
dr.narather@gmail.com

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ABSTRACT. Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to real or complex number α. If $P(z)$ does not vanish in $|z| < k, k \geq 1$, then it has been proved that for $|\alpha| \geq 1$ and $p > 0$,

$$\|D_\alpha P\|_p \leq \left(\frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p .$$

An analogous result for the class of polynomials having no zero in $|z| > k, k \leq 1$ is also obtained.

Key words and phrases: L_p inequalities, Polar derivatives, Polynomials.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P_n(z)$ denote the space of all complex polynomials $P(z)$ of degree n. For $P \in P_n$, define

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

and

$$\|P\|_\infty := \max_{|z|=1} |P(z)| .$$

If $P \in P_n$, then

(1.1) $$\|P'\|_\infty \leq n \|P\|_\infty$$

and

(1.2) $$\|P'\|_p \leq n \|P\|_p.$$

Inequality (1.1) is a well-known result of S. Bernstein (see [12] or [15]), whereas inequality (1.2) is due to Zygmund [16]. Arestov [1] proved that the inequality (1.2) remains true for

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$0 < p < 1$ as well. Equality in (1.1) and (1.2) holds for $P(z) = az^n, a \neq 0$. If we let $p \to \infty$ in (1.2), we get inequality (1.1).

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be improved. In fact, if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then (1.1) and (1.2) can be, respectively, replaced by

$$\|P'\|_\infty \leq \frac{n}{2} \|P\|_\infty \tag{1.3}$$

and

$$\|P'\|_p \leq \frac{n}{\|1+z\|_p} \|P\|_p \,, \qquad p \geq 1. \tag{1.4}$$

Inequality (1.3) was conjectured by P. Erdös and later verified by P. D. Lax [10] whereas the inequality (1.4) was discovered by De Bruijn [5]. Rahman and Schmeisser [13] proved that the inequality (1.4) remains true for $0 < p < 1$ as well. Both the estimates are sharp and equality in (1.3) and (1.4) holds for $P(z) = az^n + b$, $|a| = |b|$.

Malik [11] generalized inequality (1.3) by proving that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\|P'\|_\infty \leq \frac{n}{1+k} \|P\|_\infty \,. \tag{1.5}$$

Govil and Rahman [8] extended inequality (1.5) to the L_p-norm by proving that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|k+z\|_p} \|P\|_p \,, \quad p \geq 1. \tag{1.6}$$

It was shown by Gardner and Weems [7] and independently by Rather [14] that the inequality (1.6) remains true for $0 < p < 1$ as well.

Let $D_\alpha P(z)$ denote the polar derivative of polynomial $P(z)$ of degree n with respect to a real or complex number α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Polynomial $D_\alpha P(z)$ is of degree at most $n-1$. Furthermore, the polar derivative $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R, R > 0$.

A. Aziz [2] extended inequalities (1.1) and (1.3) to the polar derivative of a polynomial and proved that if $P \in P_n$, then for every complex number α with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq n\,|\alpha|\,\|P\|_\infty \tag{1.7}$$

and if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq \frac{n}{2}(|\alpha| + 1)\,\|P\|_\infty \,. \tag{1.8}$$

Both the inequalities (1.7) and (1.8) are sharp. If we divide both sides of (1.7) and (1.8) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequalities (1.1) and (1.3) respectively.

A. Aziz [2] also considered the class of polynomials $P \in P_n$ having no zero in $|z| < k$ and proved that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq n\left(\frac{|\alpha| + k}{1+k}\right)\|P\|_\infty \,. \tag{1.9}$$

The result is best possible and equality in (1.9) holds for $P(z) = (z + k)^n$ where α is any real number with $\alpha \geq 1$.

It is natural to seek an L_p - norm analog of the inequality (1.7). In view of the L_p - norm extension (1.2) of inequality (1.1), one would expect that if $P \in P_n$, then

$$\|D_\alpha P\|_p \leq n\,|\alpha|\,\|P\|_p\,, \tag{1.10}$$

is the L_p - norm extension of (1.7) analogous to (1.2). Unfortunately, inequality (1.10) is not, in general, true for every complex number α. To see this, we take in particular $p = 2,\ P(z) = (1 - iz)^n$ and $\alpha = i\delta$ where δ is any positive real number such that

$$1 \leq \delta < \frac{n + \sqrt{2n(2n-1)}}{3n-2}, \tag{1.11}$$

then from (1.10), by using Parseval's identity, we get, after simplication

$$n(1 + \delta)^2 \leq 2(2n - 1)\delta^2.$$

This inequality can be written as

$$\left(\delta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right)\left(\delta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) \geq 0. \tag{1.12}$$

Since $\delta \geq 1$, we have

$$\begin{aligned}\left(\delta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) &\geq \left(1 - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right)\\ &= \left(\frac{2(n-1) + \sqrt{2n(2n-1)}}{3n-2}\right) > 0\end{aligned}$$

and hence from (1.12), it follows that

$$\left(\delta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right) \geq 0.$$

This gives

$$\delta \geq \frac{n + \sqrt{2n(2n-1)}}{3n-2},$$

which clearly contradicts (1.11). Hence inequality (1.10) is not, in general, true for all polynomials of degree $n \geq 1$.

While seeking the desired extension of inequality (1.8) to the L_p-norm, recently Govil et al. [9] have made an incomplete attempt by claiming to have proved that *if $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$, and $p \geq 1$,*

$$\|D_\alpha P\|_p \leq n\left(\frac{|\alpha| + 1}{\|1 + z\|_p}\right)\|P\|_p\,. \tag{1.13}$$

A. Aziz, N.A. Rather and Q. Aliya [4] pointed out an error in the proof of inequality (1.13) given by Govil et al. [9] and proved a more general result which not only validated inequality (1.13) but also extended inequality (1.6) for the polar derivative of a polynomial $P \in P_n$. In fact, they proved that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha P\|_p \leq n\left(\frac{|\alpha| + k}{\|k + z\|_p}\right)\|P\|_p\,. \tag{1.14}$$

The main aim of this paper is to obtain certain L_p inequalities for the polar derivative of a polynomial valid for $0 < p < \infty$. We begin by proving the following extension of inequality (1.2) to the polar derivatives.

Theorem 1.1. *If $P \in P_n$, then for every complex number α and $p > 0$,*

$$\|D_\alpha P\|_p \leq n(|\alpha| + 1) \|P\|_p . \tag{1.15}$$

Remark 1. If we divide the two sides of (1.15) by $|\alpha|$ and make $|\alpha| \to \infty$, we get inequality (1.2) for each $p > 0$.

As an extension of inequality (1.6) to the polar derivative of a polynomial, we next present the following result which includes inequalities (1.13) and (1.14) for each $p > 0$ as a special cases.

Theorem 1.2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$ and $p > 0$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p . \tag{1.16}$$

In the limiting case, when $p \to \infty$, the above inequality is sharp and equality in (1.16) holds for $P(z) = (z + k)^n$ where α is any real number with $\alpha \geq 1$.

The following result immediately follows from Theorem 1.2 by taking $k = 1$.

Corollary 1.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $p > 0$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p . \tag{1.17}$$

Remark 2. Corollary 1.3 not only validates inequality (1.13) for $p \geq 1$ but also extends it for $0 < p < 1$ as well.

Remark 3. If we let $p \to \infty$ in (1.16), we get inequality (1.9). Moreover, inequality (1.6) also follows from Theorem 1.2 by dividing the two sides of inequality (1.16) by $|\alpha|$ and then letting $|\alpha| \to \infty$.

We also prove:

Theorem 1.4. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $P(0) \neq 0$, then for every complex number α with $|\alpha| \leq 1$ and $p > 0$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p . \tag{1.18}$$

In the limiting case, when $p \to \infty$, the above inequality is sharp and equality in (1.18) holds for $P(z) = (z + k)^n$ for any real α with $0 \leq \alpha \leq 1$.

The following result is an immediate consequence of Theorem 1.4.

Corollary 1.5. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for every complex number α with $|\alpha| \leq 1$,*

$$\|D_\alpha P\|_\infty \leq n \left(\frac{|\alpha| + k}{1 + k} \right) \|P\|_\infty .$$

The result is best possible and equality in (1.18) holds for $P(z) = (z + k)^n$ for any real α with $0 \leq \alpha \leq 1$.

Finally, we prove the following result.

Theorem 1.6. *If $P \in P_n$ is self- inversive, then for every complex number α and $p > 0$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p .$$

The above inequality extends a result due to Dewan and Govil [6] *for the polar derivatives.*

2. LEMMAS

For the proof of these theorems, we need the following lemmas.

Lemma 2.1 ([2])**.** *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for every real or complex number γ with $|\gamma| \geq 1$,*

$$|D_{\gamma k} P(z)| \leq k \left| D_{\gamma/k} Q(z) \right| \quad \textit{for} \quad |z| = 1$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Setting $\alpha = \gamma k$ where $k \geq 1$ in Lemma 2.1, we immediately get:

Lemma 2.2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \leq k \left| D_{\alpha/k^2} Q(z) \right| \quad \textit{for} \quad |z| = 1$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 2.3. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $|z| = 1$,*

$$k \left| P'(z) \right| \leq \left| Q'(z) \right| .$$

Lemma 2.3 is due to Malik [9].

Lemma 2.4. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for every real $\beta, 0 \leq \beta < 2\pi$,*

$$\left| k^2 P'(z) + e^{i\beta} Q'(z) \right| \leq k \left| P'(z) + e^{i\beta} Q'(z) \right| \quad \textit{for} \quad |z| = 1.$$

Proof of Lemma 2.4. By hypothesis, $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. Therefore, by Lemma 2.3, we have

$$k^2 \left| P'(z) \right|^2 \leq \left| Q'(z) \right|^2 \quad \text{for} \quad |z| = 1.$$

Multiplying both sides of this inequality by $(k^2 - 1)$ and rearranging the terms, we get

$$k^4 \left| P'(z) \right|^2 + \left| Q'(z) \right|^2 \leq k^2 \left| P'(z) \right|^2 + k^2 \left| Q'(z) \right|^2 \quad \text{for} \quad |z| = 1. \tag{2.1}$$

Adding $2 \operatorname{Re} \left(k^2 P'(z) \overline{Q'(z) e^{i\beta}} \right)$ to the both sides of (2.1), we obtain for $|z| = 1$,

$$\left| k^2 P'(z) + e^{i\beta} Q'(z) \right|^2 \leq k^2 \left| P'(z) + e^{i\beta} Q'(z) \right|^2 \quad \text{for} \quad |z| = 1$$

and hence

$$\left| k^2 P'(z) + e^{i\beta} Q'(z) \right| \leq k \left| P'(z) + e^{i\beta} Q'(z) \right| \quad \text{for} \quad |z| = 1.$$

This proves Lemma 2.4. □

Lemma 2.5. *If $P \in P_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $p > 0$ and β real, $0 \le \beta < 2\pi$,*

$$\int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta \le 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Lemma 2.5 is due to the author [14] (see also [3]).

Lemma 2.6. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \ge 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every complex number α, β real, $0 \le \beta < 2\pi$, and $p > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} \left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right|^p d\theta d\beta \le 2\pi n^p \left(|\alpha| + k\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Proof of Lemma 2.6. We have $Q(z) = z^n\overline{P(1/\overline{z})}$, therefore, $P(z) = z^n\overline{Q(1/\overline{z})}$ and it can be easily verified that for $0 \le \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} \text{ and } nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}.$$

Also, since $P \in P_n$ *and* $P(z)$ does not vanish in $|z| < k, k \ge 1$, therefore, $Q \in P_n$. Hence for every complex number α, β real, $0 \le \beta < 2\pi$, we have

$$\begin{aligned}
&\left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right| \\
&= \left|(nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + k^2 e^{i\beta}\left(nQ(e^{i\theta}) + \left(\frac{\alpha}{k^2} - e^{i\theta}\right)Q'(e^{i\theta})\right)\right| \\
&= \left|(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right) + k^2 e^{i\beta}\left(nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\right) \\
&\qquad + \alpha\left(P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right)| \\
&= \left|\left(e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} + k^2 e^{i\beta}e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\right) + \alpha\left(P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right)\right| \\
&\le |\alpha|\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right| + \left|k^2P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|.
\end{aligned}$$

This gives, with the help of Lemma 2.4,

$$\begin{aligned}
\left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right| &\le |\alpha|\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right| \\
&\qquad + k\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right| \\
&= (|\alpha| + k)\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|,
\end{aligned}$$

which implies for each $p > 0$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right|^p d\theta d\beta \\
&\qquad \le (|\alpha| + k)^p \int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta.
\end{aligned}$$

Combining this with Lemma 2.5, we get

$$\int_0^{2\pi}\int_0^{2\pi} \left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right|^p d\theta d\beta \le 2\pi n^p \left(|\alpha| + k\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

This completes the proof of Lemma 2.6. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. Let $Q(z) = z^n\overline{P(1/\overline{z})}$, then $P(z) = z^n\overline{Q(1/\overline{z})}$ and (as before) for $0 \le \theta < 2\pi$, we have

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} \qquad \text{and} \qquad nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})},$$

which implies for every complex number α and β real, $0 \le \beta < 2\pi$,

$$\begin{aligned}
&\left|D_\alpha P(e^{i\theta}) + e^{i\beta}\left\{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\right\}\right| \\
&= |nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + e^{i\beta}\left\{nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) + \alpha Q'(e^{i\theta})\right\}| \\
&= |\left\{nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right\} + e^{i\beta}\left\{nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\right\} \\
&\qquad + \alpha\left\{P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right\}| \\
&= |e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} + e^{i\beta}e^{i(n-1)\theta}\overline{P'e^{i\theta})} + \alpha\left\{P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right\}| \\
&\le |e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} + e^{i\beta}e^{i(n-1)\theta}\overline{\prime(e^{i\theta})}| + |\alpha|\ |P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})| \\
&= (|\alpha| + 1)\ |P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})|.
\end{aligned}$$

This gives with the help of Lemma 2.5 for each $p > 0$,

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi} &\left|D_\alpha P(e^{i\theta}) + e^{i\beta}\left\{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\right\}\right|^p d\theta d\beta \\
&\le (|\alpha| + 1)^p \int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta \\
&\le 2\pi n^p (|\alpha| + 1)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta. \qquad (3.1)
\end{aligned}$$

Now using the fact that for any $p > 0$,

$$\int_0^{2\pi} \left|a + be^{i\beta}\right|^p d\beta \ge 2\pi \max\left(|a|^p, |b|^p\right),$$

(see [5, Inequality (2.1)]), it follows from (3.1) that

$$\left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \le n\left(|\alpha| + 1\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}, \quad p > 0.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Since $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \ge 1$, by Lemma 2.2, we have for every real or complex number α with $|\alpha| \ge 1$,

$$|D_\alpha P(z)| \le k\left|D_{\alpha/k^2}Q(z)\right| \qquad \text{for} \quad |z| = 1, \qquad (3.2)$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Also, by Lemma 2.6, for every real or complex number $\alpha,\ p > 0$ and β real,

$$\begin{aligned}
\int_0^{2\pi}\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta}) + e^{i\beta}k^2 D_{\alpha/k^2}Q(e^{i\theta})\right|^p d\beta\right\} d\theta \qquad (3.3)\\
\le 2\pi n^p (|\alpha| + k)^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta.
\end{aligned}$$

Now for every real $\beta, 0 \le \beta < 2\pi$ and $R \ge r \ge 1$, we have

$$\left|R + e^{i\beta}\right| \ge \left|r + e^{i\beta}\right|,$$

which implies

$$\int_0^{2\pi} \left|R + e^{i\beta}\right|^p d\beta \geq \int_0^{2\pi} \left|r + e^{i\beta}\right|^p d\beta, \qquad p > 0.$$

If $D_\alpha P(e^{i\theta}) \neq 0$, we take $R = k^2 \left|D_{\alpha/k^2} Q(e^{i\theta})\right| / \left|D_\alpha P(e^{i\theta})\right|$ and $r = k$, then by (3.2), $R \geq r \geq 1$, and we get

$$\begin{aligned}
\int_0^{2\pi} &|D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\beta \\
&= \left|D_\alpha P(e^{i\theta})\right|^p \int_0^{2\pi} \left|\frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} e^{i\beta} + 1\right|^p d\beta \\
&= \left|D_\alpha P(e^{i\theta})\right|^p \int_0^{2\pi} \left|\left|\frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right| e^{i\beta} + 1\right|^p d\beta \\
&= \left|D_\alpha P(e^{i\theta})\right|^p \int_0^{2\pi} \left|\left|\frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right| + e^{i\beta}\right|^p d\beta \\
&\geq \left|D_\alpha P(e^{i\theta})\right|^p \int_0^{2\pi} \left|k + e^{i\beta}\right|^p d\beta.
\end{aligned}$$

For $D_\alpha P(e^{i\theta}) = 0$, this inequality is trivially true. Using this in (3.3), we conclude that for every real or complex number α with $|\alpha| \geq 1$ and $p > 0$,

$$\int_0^{2\pi} \left|k + e^{i\beta}\right|^p d\beta \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^p d\theta \leq 2\pi n^p \left(|\alpha| + k\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,$$

which immediately leads to (1.16) and this completes the proof of Theorem 1.2. □

Proof of Theorem 1.4. By hypothesis, all the zeros of polynomial $P(z)$ of degree n lie in $|z| \leq k$ where $k \leq 1$ and $P(0) \neq 0$. Therefore, if $Q(z) = z^n \overline{P(1/\overline{z})}$, then $Q(z)$ is a polynomial of degree n which does not vanish in $|z| < (1/k)$, where $(1/k) \geq 1$. Applying Theorem 1.2 to the polynomial $Q(z)$, we get for every real or complex number β with $|\beta| \geq 1$ and $p > 0$,

$$\left\{\int_0^{2\pi} \left|D_\beta Q(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq n \left(\frac{|\beta| + \frac{1}{k}}{\left\|z + \frac{1}{k}\right\|_p}\right) \left\{\int_0^{2\pi} \left|Q(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{3.4}$$

Now since

$$\left|Q(e^{i\theta})\right| = \left|P(e^{i\theta})\right|, \qquad 0 \leq \theta < 2\pi$$

and

$$\left\|z + \frac{1}{k}\right\|_p = \frac{1}{k} \left\|z + k\right\|_p,$$

it follows that (3.4) is equivalent to

$$\left\{\int_0^{2\pi} \left|D_\beta Q(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq n \left(\frac{k\,|\beta| + 1}{\left\|z + k\right\|_p}\right) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{3.5}$$

Also, we have for every β with $|\beta| \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}\left|D_\beta Q(e^{i\theta})\right| &= \left|nQ(e^{i\theta}) + (\beta - e^{i\theta})Q'(e^{i\theta})\right| \\ &= \left|ne^{in\theta}\overline{P(e^{i\theta})} + (\beta - e^{i\theta})\left(ne^{i(n-1)\theta}\overline{P(e^{i\theta})} - e^{i(n-2)\theta}\overline{P'(e^{i\theta})}\right)\right| \\ &= \left|\beta\left(n\overline{P(e^{i\theta})} - \overline{e^{i\theta}P'(e^{i\theta})}\right) + \overline{P'(e^{i\theta}))}\right| \\ &= \left|\overline{\beta}\left(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right) + P'(e^{i\theta})\right| \\ &= \left|\overline{\beta}\right|\left|D_{1/\overline{\beta}}P(e^{i\theta})\right|.\end{aligned}$$

Using this in (3.5), we get for $|\beta| \geq 1$,

$$\left\{\int_0^{2\pi} |\beta|\left|D_{1/\overline{\beta}}P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq n\left(\frac{k\,|\beta| + 1}{\|z+k\|_p}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}, \quad p > 0. \tag{3.6}$$

Replacing $1/\overline{\beta}$ by α so that $|\alpha| \leq 1$, we obtain from (3.6)

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq n\left(\frac{|\alpha| + k}{\|z+k\|_p}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}},$$

for $|\alpha| \leq 1$ and $p > 0$. This proves Theorem 1.4. □

Proof of Theorem 1.6. Since $P(z)$ is a self inversive polynomial of degree n, $P(z) = Q(z)$ for all $z \in \mathbb{C}$ where $Q(z) = z^n\overline{P(1/\overline{z})}$. This gives for every complex number α,

$$|D_\alpha P(z)| = |D_\alpha Q(z)|, \quad z \in \mathbb{C}$$

so that

$$\left|D_\alpha Q(e^{i\theta})/D_\alpha P(e^{i\theta})\right| = 1, \quad 0 \leq \theta < 2\pi. \tag{3.7}$$

Also, since $Q(z)$ is a polynomial of degree n, then

$$D_\alpha Q(e^{i\theta}) = nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) + \alpha Q'(e^{i\theta}). \tag{3.8}$$

Combining (3.1) and (3.8), it follows that for every complex number α and $p > 0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta}) + D_\alpha Q(e^{i\theta})\right|^p d\theta d\beta \;\leq 2\pi n^p\left(|\alpha| + 1\right)^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \tag{3.9}$$

Using (3.7) in (3.9) and proceeding similarly as in the proof of Theorem 1.2, we immediately get the conclusion of Theorem 1.6. □

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