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ON THE COMPOSITION OF SOME ARITHMETIC FUNCTIONS, II

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ABSTRACT. We study certain properties and conjuctures on the composition of the arithmetic functions σ, φ, ψ, where σ is the sum of divisors function, φ is Euler's totient, and ψ is Dedekind's function.

Key words and phrases: Arithmetic functions, Makowski-Schinzel conjuncture, Sándor's conjuncture, Inequalities.

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1. INTRODUCTION

Let $\sigma(n)$ denote the sum of divisors of the positive integer n, i.e. $\sigma(n) = \sum_{d/n} d$, where by convention $\sigma(1) = 1$. It is well-known that n is called *perfect* if $\sigma(n) = 2n$. Euclid and Euler ([10], [21]) have determined all even perfect numbers, by showing that they are of the form $n = 2^k(2^{k+1} - 1)$, where $2^{k+1} - 1$ is a prime ($k \geq 1$). The primes of the form $2^{k+1} - 1$ are the so-called Mersenne primes, and at this moment there are known exactly 41 such primes (for the recent discovery of the 41^{th} Mersenne prime, see the site *www.ams.org*). It is possible that there are infinitely many Mersenne primes, but the proof of this result seems unattackable at present. On the other hand, no odd perfect number is known, and the existence of such numbers is one of the most difficult open problems of Mathematics. D. Suryanarayana [23] defined the notion of a *superperfect* number, i.e. a number n with the property $\sigma(\sigma(n)) = 2n$, and he and H.J. Kanold [23], [11] have obtained the general form of even superperfect numbers. These are $n = 2^k$, where $2^{k+1} - 1$ is a prime. Numbers n with the property $\sigma(n) = 2n - 1$ have been called *almost perfect*, while that of $\sigma(n) = 2n + 1$, *quasi-perfect*. For many results and conjectures on this topic, see [9], and the author's book [21] (see Chapter 1).

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For an arithmetic function f, the number n is called *f-perfect*, if $f(n) = 2n$. Thus, the superperfect numbers will be in fact the $\sigma \circ \sigma$-perfect numbers where "$\circ$" denotes composition.

The Euler totient function, resp. Dedekind's arithmetic function are given by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \qquad \psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right), \tag{1.1}$$

where p runs through the distinct prime divisors of n. Following convention we let, $\varphi(1) = 1, \psi(1) = 1$. All these functions are multiplicative, i.e. they satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m,n) = 1$. For results on $\psi \circ \psi$-perfect, $\psi \circ \sigma$-perfect, $\sigma \circ \psi$-perfect, and $\psi \circ \varphi$-perfect numbers, see the first part of [18]. Let $\sigma^*(n)$ be the sum of unitary divisors of n, given by

$$\sigma^*(n) = \prod_{p^\alpha || n} (p^\alpha + 1), \tag{1.2}$$

where $p^\alpha || n$ means that for the prime power p^α one has $p^\alpha | n$, but $p^{\alpha+1} \nmid n$. By convention, let $\sigma^*(1) = 1$. In [18] almost and quasi $\sigma^* \circ \sigma^*$-perfect numbers (i.e. satisfying $\sigma^*(\sigma^*(n)) = 2n \mp 1$) are studied, where it is shown that for $n > 3$ there are no such numbers. This result has been rediscovered by V. Sitaramaiah and M.V. Subbarao [22].

In 1964, A. Makowski and A. Schinzel [13] conjectured that

$$\sigma(\varphi(n)) \geq \frac{n}{2}, \text{ for all } n \geq 1. \tag{1.3}$$

The first results after the Makowski and Schinzel paper were proved by J. Sándor [16], [17]. He proved that (1.3) holds if and only if

$$\sigma(\varphi(m)) \geq m, \text{ for all odd } m \geq 1 \tag{1.4}$$

and obtained a class of numbers satisfying (1.3) and (1.4). But (1.4) holds iff is it true for squarefree n, see [17], [18]. This has been rediscovered by G.L. Cohen and R. Gupta ([4]). Many other partial results have been discovered by C. Pomerance [14], G.L. Cohen [4], A. Grytczuk, F. Luca and M. Wojtowicz [7], [8], F. Luca and C. Pomerance [12], K. Ford [6]. See also [2], [19], [20]. Kevin Ford proved that

$$\sigma(\varphi(n)) \geq \frac{n}{39.4}, \text{ for all } n. \tag{1.5}$$

In 1988 J. Sándor [15], [16] conjectured that

$$\psi(\varphi(m)) \geq m, \text{ for all odd } m. \tag{1.6}$$

He showed that (1.6) is equivalent to

$$\psi(\varphi(n)) \geq \frac{n}{2} \tag{1.7}$$

for all n, and obtained a class of numbers satisfying these inequalities. In 1988 J. Sándor [15] conjectured also that

$$\varphi(\psi(n)) \leq n, \text{ for any } n \geq 2 \tag{1.8}$$

and V. Vitek [24] of Praha verified this conjecture for $n \leq 10^4$.

In 1990 P. Erdős [5] expressed his opinion that this new conjecture could be as difficult as the Makowski-Schinzel conjecture (1.3). In 1992 K. Atanassov [3] believed that he obtained a proof of (1.8), but his proof was valid only for certain special values of n.

Nonetheless, as we will see, conjectures (1.6), (1.7) and (1.8) are not generally true, and it will be interesting to study the classes of numbers for which this is valid.

The aim of this paper is to study this conjecture and certain new properties of the above – and related – composite functions.

1.1. Basic symbols and notations.

- $\sigma(n)$ = sum of divisors of n,
- $\sigma^*(n)$ = sum of unitary divisors of n,
- $\varphi(n)$ = Euler's totient function,
- $\psi(n)$ = Dedekind's arithmetic function,
- $[x]$ = integer part of x,
- $\omega(n)$ = number of distinct divisors of n,
- $a|b$ = a divides b,
- $a \nmid b$ = a does not divide b,
- $pr\{n\}$ = set of distinct prime divisors of n,
- $f \circ g$ = composition of f and g.

2. BASIC LEMMAS

Lemma 2.1.

$$\varphi(ab) \le a\varphi(b), \text{ for any } a, b \ge 2 \tag{2.1}$$

with equality only if $pr\{a\} \subset pr\{b\}$, *where* $pr\{a\}$ *denotes the set of distinct prime factors of* a.

Proof. We have

$$ab = \prod_{p|a, p\nmid b} p^{\alpha} \cdot \prod_{q|a, q|b} q^{\beta} \cdot \prod_{r|b, r\nmid a} r^{\gamma},$$

so

$$\begin{aligned}\frac{\varphi(ab)}{ab} &= \prod \left(1 - \frac{1}{p}\right) \cdot \prod \left(1 - \frac{1}{q}\right) \cdot \prod \left(1 - \frac{1}{r}\right)\\ &\le \prod \left(1 - \frac{1}{q}\right) \cdot \prod \left(1 - \frac{1}{r}\right) = \frac{\varphi(b)}{b},\end{aligned}$$

so $\varphi(ab) \le a\varphi(b)$, with equality if "$p$ does not exist", i.e. p with the property $p|a, p \nmid b$. Thus for all $p|a$ one has also $p|b$. □

Lemma 2.2. *If* $pr\{a\} \not\subset pr\{b\}$, *then for any* $a, b \ge 2$ *one has*

$$\varphi(ab) \le (a-1)\varphi(b), \tag{2.2}$$

and

$$\psi(ab) \ge (a+1)\psi(b). \tag{2.3}$$

Proof. We give only the proof of (2.2).

Let $a = \prod p^{\alpha} \cdot \prod q^{\beta}$, $b = \prod r^{\gamma} \cdot \prod q^{\beta'}$, where the q are the common prime factors, and the $p \in pr\{a\}$ are such that $p \notin pr\{b\}$, i.e. suppose that $\alpha \ge 1$. Clearly $\beta, \beta', \gamma \ge 0$. Then

$$\frac{\varphi(ab)}{\varphi(b)} = a \cdot \prod \left(1 - \frac{1}{p}\right) \le a - 1$$

iff

$$\prod \left(1 - \frac{1}{p}\right) \le 1 - \frac{1}{a} = 1 - \frac{1}{\prod p^{\alpha} \cdot \prod q^{\beta}}.$$

Now,

$$1 - \frac{1}{\prod p^{\alpha} \cdot \prod q^{\beta}} \ge 1 - \frac{1}{\prod p^{\alpha}} \ge 1 - \frac{1}{\prod p}$$

by $\alpha \geq 1$. The inequality

$$1 - \frac{1}{\prod p} \geq \prod \left(1 - \frac{1}{p}\right)$$

is trivial, since by putting e.g. $p - 1 = u$, one gets

$$\prod (u+1) \geq 1 + \prod u,$$

and this is clear, since $u > 0$. There is equality only when there is a single u, i.e. if the set of p such that $pr\{a\} \not\subset pr\{b\}$ has a single element, at the first power, and all $\beta = 0$, i.e. when $a = p \nmid b$. Indeed:

$$\varphi(pb) = \varphi(p)\varphi(b) = (p-1)\varphi(b).$$

□

Lemma 2.3. *For all $a, b \geq 1$,*

$$\sigma(ab) \geq a\sigma(b), \tag{2.4}$$

and

$$\psi(ab) \geq a\psi(b). \tag{2.5}$$

Proof. (2.4) is well-known, see e.g. [16], [18]. There is equality here, only for $a = 1$.

For (2.5), let $u|v$. Then

$$\frac{\psi(u)}{u} = \prod_{p|u}\left(1 + \frac{1}{p}\right) \leq \prod_{p|v, p|u}\left(1 + \frac{1}{p}\right) \cdot \prod_{q|v, q\nmid u}\left(1 + \frac{1}{q}\right) = \frac{\psi(v)}{v},$$

with equality if q does not exist with $q|v, q \nmid v$. Put $v = ab$ and $u = b$. Then $\frac{\psi(u)}{u} \leq \frac{\psi(v)}{v}$ becomes exactly (2.5). There is equality if for each $p|a$ one also has $p|b$, i.e. $pr\{a\} \subset pr\{b\}$. □

Remark 2.4. Therefore, there is a similarity between the inequalities (2.1) and (2.5).

Lemma 2.5. *If $pr\{a\} \not\subset pr\{b\}$, then for any $a, b \geq 2$ one has*

$$\sigma(ab) \geq \psi(a) \cdot \sigma(b). \tag{2.6}$$

Proof. This is given in [16]. □

3. MAIN RESULTS

Theorem 3.1. *There are infinitely many n such that*

$$\psi(\varphi(n)) < \varphi(\psi(n)) < n. \tag{3.1}$$

For infinitely many m one has

$$\varphi(\psi(m)) < \psi(\varphi(m)) < m. \tag{3.2}$$

There are infinitely many k such that

$$\varphi(\psi(k)) = \frac{1}{2}\psi(\varphi(k)). \tag{3.3}$$

Proof. We prove that (3.1) is valid for $n = 3 \cdot 2^a$ for any $a \geq 1$. This follows from $\varphi(3 \cdot 2^a) = 2^a$, $\psi(2^a) = 3 \cdot 2^{a-1}$, $\psi(3 \cdot 2^a) = 3 \cdot 2^{a+1}$, $\varphi(3 \cdot 2^{a+1}) = 2^{a+1}$, so

$$3 \cdot 2^a > \varphi(\psi(3 \cdot 2^a)) > \psi(\varphi(3 \cdot 2^a)).$$

For the proof of (3.2), put $m = 2^a \cdot 5^b$ $(b \geq 2)$. Then an easy computation shows that $\psi(\varphi(m)) = 2^{a+1} \cdot 3^2 \cdot 5^{b-2}$, and $\varphi(\psi(m)) = 2^{a+2} \cdot 3 \cdot 5^{b-2}$ and the inequalities (3.2) will follow.

For $h = 3^s$ remark that $\varphi(\psi(h)) = \frac{4}{9} \cdot h$ and $\psi(\varphi(h)) = \frac{4}{3} \cdot h$, so

$$\varphi(\psi(h)) < h < \psi(\varphi(h)), \tag{3.4}$$

which complete (3.1) and (3.2), in a certain sense.

Finally, for $k = 2^a \cdot 7^b$ $(b \geq 2)$ one can deduce $\psi(\varphi(k)) = \frac{48}{49} \cdot k$, $\varphi(\psi(k)) = \frac{24}{49} \cdot k$, so (3.3) follows. We remark that since

$$\psi(\varphi(k)) < k, \tag{3.5}$$

by (3.3) and (3.5) one can say that

$$\varphi(\psi(k)) < \frac{k}{2}, \tag{3.6}$$

for the above values of k. Remark also that for h in (3.4) one has

$$\varphi(\psi(h)) = \frac{1}{3}\psi(\varphi(h)). \tag{3.7}$$

For the values m given by (3.2) one has

$$\varphi(\psi(m)) = \frac{2}{3}\psi(\varphi(m)). \tag{3.8}$$

For $n = 2^a \cdot 3^b$ $(b \geq 2)$ one can remark that $\varphi(\psi(n)) = \psi(\varphi(n))$. $\square$

More generally, one can prove:

Theorem 3.2. *Let $1 < n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ the prime factorization of n and suppose that the odd part of n is squarefull, i.e. $\alpha_i \geq 2$ for all i with $p_i \geq 3$.*

Then $\varphi(\psi(n)) = \psi(\varphi(n))$ is true if and only if

$$\begin{aligned} &pr\{(p_1 - 1) \cdots (p_r - 1)\} \subset pr\{p_1, \dots, p_r\} \quad \textit{and} \\ &pr\{(p_1 + 1) \cdots (p_r + 1)\} \subset pr\{p_1, \dots, p_r\}. \end{aligned} \tag{3.9}$$

Proof. Since

$$\varphi(n) = p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 - 1) \cdots (p_r - 1)$$

and

$$\psi(n) = p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 + 1) \cdots (p_r + 1),$$

one can write

$$\psi(\varphi(n)) = p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 - 1) \cdots (p_r - 1) \cdot \prod_{t | (p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 - 1) \cdots (p_r - 1))} 1 + \frac{1}{t}$$

and

$$\varphi(\psi(n)) = p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 + 1) \cdots (p_r + 1) \cdot \prod_{q | (p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} \cdot (p_1 + 1) \cdots (p_r - 1))} \left(1 - \frac{1}{q}\right).$$

Since $\alpha_i - 1 \geq 1$ when $p_i \geq 3$, the equality $\psi(\varphi(n)) = \varphi(\psi(n))$, by

$$\begin{aligned} &(p_1 - 1) \cdots (p_r - 1) \cdot \left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_r}\right) \\ &\qquad = (p_1 + 1) \cdots (p_r + 1) \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right), \end{aligned}$$

can also be written as

$$\prod_{t | (p_1 - 1) \cdots (p_r - 1)} \left(1 + \frac{1}{t}\right) = \prod_{q | (p_1 + 1) \cdots (p_r + 1)} \left(1 - \frac{1}{q}\right).$$

Since $1+\frac{1}{t}>1$ and $1-\frac{1}{q}<1$, this is impossible in general. It is possible only if all prime factors of $(p_1+1)\cdots(p_r-1)$ are among $p_1,\dots,p_r$, and also the same for the prime factors of $(p_1+1)\cdots(p_r+1)$. □

Remark 3.3. For example, $n=2^a\cdot 3^b\cdot 5^c$ with $a\geq 1$, $b\geq 2$, $c\geq 2$ satisfy (3.9). Indeed

$$pr\{(2-1)(3-1)(5-1)\}=\{2\}, pr\{(2+1)(3+1)(5+1)\}=\{2,3\}.$$

Similar examples are $n=2^a\cdot 3^b\cdot 5^c\cdot 7^d$, $n=2^a\cdot 3^b\cdot 5^c\cdot 11^d$, $n=2^a\cdot 3^b\cdot 7^c\cdot 13^d$, $n=2^a\cdot 3^b\cdot 5^c\cdot 7^d\cdot 11^e\cdot 13^f$, $n=2^a\cdot 3^b\cdot 17^c$, etc.

Theorem 3.4. *Let n be squarefull. Then inequality (1.8) is true.*

Proof. Let $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ with $\alpha_i\geq 2$ for all $i=\overline{1,r}$. Then

$$\begin{aligned}\varphi(\psi(n))&=\varphi(p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1}\cdot(p_1+1)\cdots(p_r+1))\\&\leq(p_1+1)\cdots(p_r+1)\cdot\varphi(p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1}),\end{aligned}$$

by Lemma 2.1. But

$$\varphi(p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1})=p_1^{\alpha_1-2}\cdots p_r^{\alpha_r-2}\cdot(p_1-1)\cdots(p_r-1),$$

since $\alpha\geq 2$. Then

$$\begin{aligned}\varphi(\psi(n))&\leq(p_1^2-1)\cdots(p_r^2-1)\cdot p_1^{\alpha_1-2}\cdots p_r^{\alpha_r-2}\\&=p_1^{\alpha_1}\cdots p_r^{\alpha_r}\cdot\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right),\end{aligned}$$

so

$$\varphi(\psi(n))\leq n\cdot\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right). \tag{3.10}$$

There is equality in (3.10) if

$$pr\{(p_1+1)\cdots(p_r+1)\}\subset\{p_1,\dots,p_r\}.$$

Clearly, inequality (3.10) is best possible, and by

$$\left(1-\frac{1}{p_1^2}\right)\cdots\left(1-\frac{1}{p_r^2}\right)<1,$$

it implies inequality (1.8). □

Theorem 3.5. *For any $n\geq 2$ one has*

$$\varphi\left(n\left[\frac{\psi(n)}{n}\right]\right)<n, \tag{3.11}$$

where $[x]$ denotes the integer part of x.

Proof. It is immediate that

$$\frac{\varphi(n)\psi(n)}{n^2}=\prod_{p|n}\left(1-\frac{1}{p^2}\right)<1,$$

so $\varphi(n)\psi(n)<n^2$ for any $n\geq 2$. Now, by (2.1) one can write

$$\varphi\left(n\left[\frac{\psi(n)}{n}\right]\right)\leq\left[\frac{\psi(n)}{n}\right]\varphi(n)\leq\frac{\psi(n)}{n}\cdot\varphi(n)<n,$$

by the relation proved above. □

Remark 3.6. If $n|\psi(n)$, i.e., when $\left[\frac{\psi(n)}{n}\right] = \frac{\psi(n)}{n}$, relation (3.11) gives inequality (1.8), i.e. $\varphi(\psi(n)) < n$. For the study of an equation

$$\psi(n) = k \cdot n \tag{3.12}$$

we shall use a notion and a method of Ch. Wall [25]. We say that n is ω-multiple of m if $m|n$ and $pr\{m\} = pr\{n\}$.

We need a simple result, stated as:

Lemma 3.7. *If m and n are squarefree, and $\frac{\psi(n)}{n} = \frac{\psi(m)}{m}$, then $n = m$.*

Proof. Without loss of generality we may suppose

$$(m,n) = 1;\ m,n > 1,\ m = q_1 \cdots q_j\ (q_1 < \cdots < q_j)$$

and

$$n = p_1 \cdots p_k\ (p_1 < \cdots < p_k).$$

Then the assumed equality has the form

$$n(1+q_1)\cdots(1+q_j) = m(1+p_1)\cdots(1+p_k).$$

Since $p_k|n$, the relation

$$p_k|(1+p_1)\cdots(1+p_{k-1})(1+p_k)$$

implies $p_k|(1+p_k)$ for some $i \in \{1,2,\ldots,k\}$. Here

$$1+p_1 < \cdots < 1+p_{k-1} < 1+p_k,$$

so we must have $p_k|(1+p_{k-1})$. This may happen only when $k = 2,\ p_1 = 2,\ p_2 = 3;\ j = 2,\ q_1 = 2,\ q_3 = 3$ (since for $k \geq 3,\ p_k - p_{k-1} \geq 2$, so $p_k \nmid (1+p_{k-1})$). In this case $(n,m) = 6 > 1$, a contradiction. Thus $k = j$ and $p_k = q_j$. □

Theorem 3.8. *Assume that the least solution n_k of (3.12) is a squarefree number. Then all solutions of (3.12) are given by the ω-multiples of n_k.*

Proof. If n is ω-multiple of n_k, then clearly

$$\frac{\psi(n)}{n} = \frac{\psi(n_k)}{n_k} = k,$$

by (1.1).Conversely, if n is a solution, set $m =$ greatest squarefree divisor of n. Then

$$\frac{\psi(n)}{n} = \frac{\psi(m)}{m} = k = \frac{\psi(n_k)}{n_k}.$$

By Lemma 3.7, $m = n_k$, i.e. n is an ω-multiple of n_k. □

Theorem 3.9. *Let $n \geq 3$, and suppose that n is ψ-deficient, i.e. $\psi(n) < 2n$. Then inequality (1.8) holds.*

Proof. First remark that for any $n \geq 3,\ \psi(n)$ is an even number. Indeed, if $n = 2^a$, then $\psi(n) = 2^{a-1} \cdot 3$, which is odd only for $a = 1$, i.e. $n = 2$. If n has at least one odd prime factor p, then by (1.1), $\psi(n)$ will be even.

Now, applying Lemma 2.1 for $b = 2$, one obtains $\varphi(2a) \leq a$, i.e. $\varphi(u) \leq \frac{u}{2}$ for $u = 2a$ (even). Here equality occurs only when $u = 2^k\ (k \geq 1)$. Now, $\varphi(\psi(n)) \leq \frac{\psi(n)}{2}$, $\psi(n)$ being even, and since n is ψ-deficient, the theorem follows. □

Remark 3.10. The inequality

$$\varphi(\psi(n)) \le \frac{\psi(n)}{2} \tag{3.13}$$

is best possible, since we have equality for $\psi(n) = 2^k$. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$; then $p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1+1) \cdots (p_r+1) = 2^k$ is possible only if $\alpha_1 = \cdots = \alpha_r = 1$, and $p_1+1 = 2_1^a, \ldots, p_r+1 = 2_r^a$; i.e. when $p_1 = 2_1^a - 1, \ldots, p_r = 2_r^a - 1$ are distinct Mersenne primes, and $n = p_1 \cdots p_r$. So, there is equality in (3.13) iff n is a product of distinct Mersenne primes. Since by Theorem 3.8 one has $\psi(n) = 2n$ iff $n = 2^a \cdot 3^b$ $(a, b \ge 1)$, if one assumes $\psi(n) \le 2n$, then by (3.13), inequality (1.8) follows again. Therefore, in Theorem 3.9 one may assume $\psi(n) \le 2n$.

Let $\omega(n)$ denote the number of distinct prime factors of n. Theorem 3.9 and the above remark implies that when n is even, and $\omega(n) \le 2$, (1.8) is true. Indeed, $1 + \frac{1}{2} = \frac{3}{2} < 2$, and $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) = 2$. So e.g. when $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$, then

$$\frac{\psi(n)}{n} = \left(1 + \frac{1}{p_1}\right) \cdot \left(1 + \frac{1}{p_2}\right) \le \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) = 2.$$

On the other hand, if n is odd, and $\omega(n) \le 4$, then (1.8) is valid. Indeed,

$$\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{11}\right) = \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} = \frac{2304}{1155} < 2.$$

Another remark is the following:

If 2 and 3 do not divide n, and n has at most six prime factors, then $\varphi(\psi(n)) < n$. If 2, 3 and 5 do not divide n, and n has at most 12 prime factors, then the same result holds true. If 2, 3, 5 and 7 do not divide n, and n has at most 21 prime factors, then the inequality is true.

If 2 and 3 do not divide n, we prove that $\psi(n) < 2n$, and by the presented method the results will follow. E.g., when n is not divisible by 2 and 3, then the least prime factor of n could be 5, so

$$\frac{\psi(n)}{n} < \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \frac{32}{31} < 2,$$

and the first result follows. The other affirmations can be proved in a similar way.

In [16] it is proved that

$$\psi(n) \le \begin{cases} 3^{\omega(n)} \cdot \varphi(n), \text{ if n is even} \\ 2^{\omega(n)} \cdot \varphi(n), \text{ if n is odd} \end{cases}. \tag{3.14}$$

Thus, as a corollary of (3.13) and (3.14) one can state that if $\frac{3^{\omega(n)} \cdot \varphi(n)}{2} < n$ (or $\le n$), for n even; and $2^{\omega(n)-1} \cdot \varphi(n)$ (or $\le n$) for n odd, then relation (1.8) is valid.

By (3.13), if n is a product of distinct Mersenne primes, then $\varphi(\psi(n)) = \frac{\psi(n)}{2}$. We will prove that $\psi(n) < 2n$ for such n, thus obtaining:

Theorem 3.11. *If n is a product of distinct Mersenne primes, then inequality (1.8) is valid.*

Proof. Let $n = M_1 \cdots M_s$, where $M_i = 2^{p_i} - 1$ (p_i primes, $i = 1, 2, \ldots, s$) are distinct Mersenne primes. We have to prove that $(2^{p_1} - 1) \cdots (2^{p_s} - 1) > 2^{p_1 + \cdots + p_s - 1}$, or equivalently, $\left(1 - \frac{1}{2^{p_1}}\right) \cdots \left(1 - \frac{1}{2^{p_s}}\right) > \frac{1}{2}$. Clearly $p_1 \ge 2, p_2 \ge 3, \ldots, p_s \ge s+1$, so it is sufficient to prove that

$$\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{2^{s+1}}\right) > \frac{1}{2}. \tag{3.15}$$

In the proof of (3.15) we will use the classical Weierstrass inequality

$$\prod_{k=1}^{s}(1-a_k) > 1-\sum_{k=1}^{s} a_k, \tag{3.16}$$

where $a_k \in (0,1)$ (see e.g. D.S. Mitrinović: *Analytic inequalities,* Springer-Verlag, 1970).

Put $a_k = \frac{1}{2^{k+1}}$ in (3.16). Since

$$\sum_{k=1}^{s}\frac{1}{2^{k+1}} = \frac{1}{4}\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}\right) = \frac{1}{4}\cdot\left(\frac{1-\frac{1}{2^k}}{1-\frac{1}{2}}\right) = \frac{2^k-1}{2^k+1},$$

(3.15) becomes equivalent to $1-\frac{2^k-1}{2^k+1} > \frac{1}{2}$, or $\frac{1}{2} > \frac{2^k-1}{2^k+1}$, i.e. $2^k > 2^k-1$, which is true. Therefore, (3.15) follows, and the theorem is proved. □

Remark 3.12. By Theorem 3.23 (see relation (3.29)), if $n = M_1^{a_1}\cdots M_s^{s_s}$ (with arbitrary $a_i \geq 1$), the inequality (1.8) holds true.

Related to the above theorems is the following result:

Theorem 3.13. *Let n be even, and suppose that the greatest odd part m of n is ψ-deficient, and that $3 \nmid \psi(m)$. Then (1.8) is true.*

Proof. Let $n = 2^k \cdot m$, when

$$\varphi(\psi(n)) = \varphi(2^{k-1}\cdot 3\psi(m)) = 2\cdot\varphi(2^{k-1}\cdot\psi(m))$$

since $(3, 2^{k-1}\cdot\psi(m)) = 1$. But

$$\varphi(2^{k-1}\cdot\psi(m)) \leq 2^{k-2}\cdot\psi(m) < 2^{k-1}\cdot m,$$

so $\varphi(\psi(n)) < 2^k\cdot m = n.$ □

Remark 3.14. In [18] it is proved that for all $n \geq 2$ even, one has

$$\varphi(\sigma(n)) \geq 2n, \tag{3.17}$$

with equality only if $n = 2^k$, where $2^{k+1}-1 =$ prime. The proof is based on Lemma 2.3. Since $\sigma(m) \geq \psi(m)$, clearly this implies

$$\sigma(\sigma(n)) \geq 2n, \tag{3.18}$$

with the above equalities. So, the Surayanarayana-Kanved theorem is reobtained, in an improved form.

In [18] it is proved also that for all $n \geq 2$ even, one has

$$\sigma(\psi(n)) \geq 2n, \tag{3.19}$$

with equality only for $n = 2$. What are the odd solutions of $\sigma(\psi(n)) = 2n$?

We now prove:

Theorem 3.15. *Let $n = 2^k\cdot m$ be even ($k \geq 1, m > 1$ odd), and suppose that m is not a product of distinct Fermat primes, and that m satisfies (1.6). Then*

$$\sigma(\varphi(n)) \geq n-m \geq \frac{n}{2}. \tag{3.20}$$

Proof. First remark that if m is not a product of distinct Fermat primes, then $\varphi(m)$ is not a power of 2. Indeed, if $m = p_1^{a_1} \cdots p_r^{a_r}$, then

$$\varphi(m) = p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1 - 1) \cdots (p_r - 1) = 2^s$$

iff (since $p_i \geq 3$),

$$a_1 - 1 = \cdots = a_r - 1 = 0$$

and

$$p_1 - 1 = 2^{s_1}, \ldots, p_r - 1 = 2^{s_r},$$

i.e.

$$p_1 = 2^{s_1} + 1, \ldots, p_r = 2^{s_r} + 1$$

are distinct Fermat primes. Thus there exists at least an odd prime divisor of $\varphi(m)$. Now, by Lemma 2.5,

$$\sigma(\varphi(2^k \cdot m)) = \sigma(2^{k-1} \cdot \varphi(m)) \geq \psi(\varphi(m)) \cdot \sigma(2^{k-1}) \geq m \cdot (2^k - 1) = n - m,$$

by relation (1.6). The last inequality of (3.20) is trivial, since $m \leq \frac{n}{2} = 2^{k-1} \cdot m$, where $k - 1 \geq 0$. ☐

Remark 3.16. Relation (3.17) gives an improvement of (1.3) for certain values of n.

Theorem 3.17. *Let p be an odd prime. Then*

$$\varphi(\psi(p)) \leq \frac{p+1}{2}, \tag{3.21}$$

with equality only if p is a Mersenne prime, and $\psi(\varphi(p)) \geq \frac{3}{2} \cdot (p-1)$, with equality only if p is a Fermat prime.

Proof. $\psi(p) = p + 1$ and $p + 1$ being even, $\varphi(p+1) \leq \frac{p+1}{2}$, with equality only if $p + 1 = 2^k$, i.e. when $p = 2^k - 1 =$ Mersenne prime. Since $\frac{3}{2} \cdot (p-1) \geq p$, this inequality is better than (1.6) for $n = p$. Similarly, $\varphi(p) = p - 1 =$ even, so $\psi(p-1) \geq \frac{3}{2} \cdot (p-1)$, on base of the following: ☐

Lemma 3.18. *If $n \geq 2$ is even, then*

$$\psi(n) \geq \frac{3}{2} \cdot n, \tag{3.22}$$

with equality only if $n = 2^a$ (power of 2).

Proof. If $n = 2^a \cdot N$, with N odd,

$$\psi(n) = \psi(2^a) \cdot \psi(N) = 2^{a-1} \cdot 3 \cdot \psi(N) \geq 2^{a-1} \cdot 3 \cdot N = \frac{3}{2} \cdot n.$$

Equality occurs only, when $N = 1$, i.e. when $n = 2^a$. ☐

Since $p - 1 = 2^a$ implies $p = 2^a + 1 =$ Fermat prime, (3.21) is completely proved. Since $\frac{3}{2} \cdot (p-1) \geq p$, this inequality is better than (1.6) for $n = p$.

Remark 3.19. For $p \geq 5$ one has $\frac{p+1}{2} < p < \frac{3}{2} \cdot (p-1)$, so (3.21) implies, as a corollary that

$$\varphi(\psi(p)) < p < \psi(\varphi(p)), \tag{3.23}$$

for $p \geq 5$, prime.

This is related to relation (3.4). If n is even, and $n \neq 2^a$ (power of 2), then since $\psi(N) \geq N+1$, with equality only when N is a prime, (3.22) can be improved to

$$\psi(n) \geq \frac{3}{2} \cdot \left(n + \frac{n}{N}\right), \tag{3.24}$$

with equality only for $n = 2^a \cdot N$, where $N =$ prime.

Theorem 3.20. *Let $a, b \geq 1$ and suppose that $a|b$. Then $\varphi(\psi(a))|\varphi(\psi(b))$ and $\psi(\varphi(a))|\psi(\varphi(b))$. In particular, if $a|b$, then*

$$\varphi(\psi(a)) \leq \varphi(\psi(b)); \qquad \psi(\varphi(a)) \leq \psi(\varphi(b)). \tag{3.25}$$

Proof. The proof follows at once from the following:

Lemma 3.21. *If $a|b$, then*

$$\varphi(a)|\varphi(b), \tag{3.26}$$

and

$$\psi(a)|\psi(b), \tag{3.27}$$

□

Proof. This follows by (1.1), see e.g. [16], [18]. □

Now, if $a|b$, then $\psi(a)|\psi(b)$ by (3.27), so by (3.26), $\varphi(\psi(a))|\varphi(\psi(b))$. Similarly, $a|b$ implies $\varphi(a)|\varphi(b)$ by (3.26), so by (3.27), $\psi(\varphi(a))|\psi(\varphi(b))$. The inequalities in (3.22) are trivial consequences.

Remark 3.22. Let $a = p$ be a prime such that $p \nmid k$, and put $b = k^{p-1} - 1$.

By Fermat's little theorem one has $a|b$, so all results of (3.25) are correct in this case. For example, $\psi(\varphi(a)) \leq \psi(\varphi(b))$ gives, in the case of (3.25), and Theorem 3.15:

$$\psi(\varphi(k^{p-1} - 1)) \geq \psi(\varphi(p)) \geq \frac{3}{2} \cdot (p-1), \tag{3.28}$$

for any prime $p \nmid k$, and any positive integer $k > 1$.

Let $(n,k) = 1$. Then by Euler's divisibility theorem, one has similarly:

$$\psi(\varphi(k^{\varphi(n)} - 1)) \geq \psi(\varphi(n)), \tag{3.29}$$

for any positive integers $n, k > 1$ such that $(n,k) = 1$.

Let $n > 1$ be a positive integer, having as distinct prime factors $p_1, \dots, p_r$. Then, using (1.1) it is immediate that

$$\varphi(n)|\psi(n) \tag{3.30}$$

iff $(p_1 - 1)\cdots(p_r - 1)|(p_1 + 1)\cdots(p_r + 1)$. For example, (3.30) is true for $n = 2^m, n = 2^m \cdot 5^s$ $(m, s \geq 1)$, etc. Now assuming (3.30), by (3.26) one can write the following inequalities:

$$\varphi(\psi(\varphi(n))) \leq \varphi(\psi(\psi(n))) \text{ and } \psi(\varphi(\varphi(n))) \leq \psi(\varphi(\psi(n))). \tag{3.31}$$

By studying the first 100 values of n with the property (3.30), the following interesting example may be remarked: $\varphi(15) = \varphi(16) = 8$, $\psi(15) = \psi(16) = 24$ and $\varphi(15)|\psi(15)$. Similarly $\varphi(70) = \varphi(72) = 24$, $\psi(70) = \psi(72) = 144$, with $\varphi(70)|\psi(70)$.

Are there infinitely many such examples? Are there infinitely many n such that $\varphi(n) = \varphi(n+1)$ and $\psi(n) = \psi(n+1)$? Or $\varphi(n) = \varphi(n+2)$ and $\psi(n) = \psi(n+2)$?

Let $a = 8, b = \sigma(8k-1)$. Then $a|b$ (see e.g. [18] for such relations), and since $\psi(\varphi(8)) = 6$, $\varphi(\psi(8)) = 12$, by (3.25) we obtain the divisibility relations

$$6|\psi(\varphi(\sigma(8k-1))) \text{ and } 12|\varphi(\psi(\sigma(8k-1))) \tag{3.32}$$

for $k \geq 1$.

The second relation implies e.g. that if $\varphi(\psi(\sigma(n))) = 2n$, then $n \not\equiv -1 \pmod{8}$ and if $\varphi(\psi(\sigma(n))) = 4n$, then $n \not\equiv -1 \pmod{24}$.

Theorem 3.23. *Inequality (1.8) is true for an $n \geq 2$ if it is true for the squarefree part of $n \geq 2$. Inequality (1.6) is true for an odd $m \geq 3$ if it is true for the squarefree part of $m \geq 3$.*

Proof. As we have stated in the Introduction, such results were first proved by the author. We give here the proof for the sake of completeness.

Let n' be the squarefree part of n, i.e. if $n = p_1^{a_1} \cdots p_r^{a_r}$, then $n' = p_1 \cdots p_r$. Then

$$\begin{aligned}\varphi(\psi(n)) &= \varphi(p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot (p_1+1) \cdots (p_r+1)) \\ &\leq p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot \varphi((p_1+1) \cdots (p_r+1)) \\ &= \frac{n}{n'} \cdot \varphi(\psi(n')))\end{aligned}$$

by inequality (2.1).

Thus

$$\frac{\varphi(\psi(n))}{n} \leq \frac{\varphi(\psi(n'))}{n'}. \tag{3.33}$$

Therefore, if $\frac{\varphi(\psi(n'))}{n'} < 1$, then $\frac{\varphi(\psi(n))}{n} < 1$. Similarly one can prove that

$$\frac{\psi(\varphi(m))}{m} \geq \frac{\psi(\varphi(m'))}{m'}, \tag{3.34}$$

so if (1.6) is true for the squarefree part m' of m, then (1.6) is true also for m.

As a consequence, (1.8) is true for all n if and only if it is true for all squarefree n.

As we have stated in the introduction, (1.6) is not generally true for all m. Let e.g. $m = 3 \cdot F$, where $F > 3$ is a Fermat prime. Indeed, put $F = 2^k + 1$. Then $\varphi(m) = 2^{k+1}$, so

$$\psi(\varphi(m)) = 2^k \cdot 3 < 3 \cdot (2^k + 1) = 3 \cdot F = m,$$

contradicting (1.6). However, if m has the form $m = 5 \cdot F$, where $F > 5$ is again a Fermat prime, then (1.6) is valid, since in this case

$$\psi(\varphi(m)) = 6 \cdot 2^k > 5 \cdot (2^k + 1) = m.$$

□

More generally, we will prove now:

Theorem 3.24. *Let $5 \leq F_1 < \cdots < F_s$ be Fermat primes. Then inequality (1.6) is valid (with strict inequality) for $m = F_1^{a_1} \cdots F_s^{a_s}$, with arbitrary $a_i \geq 1$ $(i = \overline{1, s})$.*

Proof. Let $F_i = 1 + 2^{2^{b_i}}$ $(i \geq 1)$ be Fermat primes, where $b_1 \geq 1$. Since $b_1 < b_2 < \cdots < b_s$, clearly $b_i \geq i$ for any $i = 1, 2, \ldots, s$. By (3.34) it is sufficient to prove the result for $m' = F_1 \cdots F_s$, when (1.6) becomes, after some elementary computations:

$$\left(1 + \frac{1}{2^{2^{b_1}}}\right) \cdots \left(1 + \frac{1}{2^{2^{b_s}}}\right) \leq \frac{3}{2}. \tag{3.35}$$

We will prove that (3.35) holds with strict inequality. By the classical Weierstrass inequalities one has

$$\prod_{k=1}^{s} (1 + a_k) < \frac{1}{1 - \sum_{k=1}^{s} a_k},$$

where $a_k \in (0, 1)$.

Since $b_i \geq 1$, it is sufficient to prove that

$$\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{2^{2^s}}\right) \leq \frac{3}{2}. \tag{3.36}$$

Put $a_k = 2^{2^k}$ $(k \geq 1)$, so by the above inequality, it is sufficient to prove that

$$\sum = \frac{1}{2^{2^1}} + \frac{1}{2^{2^2}} + \cdots + \frac{1}{2^{2^s}} < \frac{1}{3}. \tag{3.37}$$

Clearly (3.37) is true for $s = 1, 2$, since $\frac{1}{4} < \frac{1}{3}$, $\frac{1}{4} + \frac{1}{16} = \frac{5}{16} < \frac{1}{3}$. Let $s \geq 3$. Then, since $2^s \geq s + 5$ for $s \geq 3$, we can write

$$\begin{aligned}\sum &\leq \frac{1}{4} + \frac{1}{16} + \frac{1}{2^8}\cdot\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-3}}\right)\\ &= \frac{5}{16} + \frac{1}{128}\cdot\left(1 - \frac{1}{2^{s-2}}\right)\\ &< \frac{5}{16} + \frac{1}{128} = \frac{41}{128} < \frac{1}{3},\end{aligned}$$

and the assertion is proved. □

Remark 3.25. By Lemma 2.2, relation (2.2) one can write successively

$$\begin{gathered}\varphi((p_1+1)(p_2+1)) \leq p_2\varphi(p_1+1) < p_1p_2, \text{ if } pr\{p_2+1\} \not\subset pr\{p_1+1\}\\ \varphi((p_1+1)(p_2+1)(p_3+1)) \leq p_3\varphi(p_1+1)(p_2+1) < p_1p_2p_3,\\ \text{if in addition } pr\{p_3+1\} \not\subset pr\{(p_1+1)(p_2+1)\}\\ \cdots\\ \varphi((p_1+1)\cdots(p_{r-1}+1)(p_r+1)) \leq p_r\varphi((p_1+1)\cdots(p_{r-1}+1)) < p_1\cdots p_r,\\ \text{if } pr\{p_r+1\} \not\subset pr\{(p_1+1)\cdots(p_{r-1}+1)\}\end{gathered} \tag{3.38}$$

is satisfied, then by Theorem 3.23, inequality (1.8) is valid.

Similarly, by using Lemma 2.2, (2.3), and Theorem 3.23, we can state that if

$$\begin{gathered}pr\{p_2-1\} \not\subset pr\{q_1-1\},\\ pr\{q_3-1\} \not\subset pr\{(p_1-1)(p_2-1)\},\\ \cdots,\\ pr\{q_r-1\} \not\subset pr\{(p_1-1)\cdots(q_{r-1}-1)\},\end{gathered} \tag{3.39}$$

then inequality (1.6) is valid. (Here q_1, q_2, $\ldots$, q_r are the prime divisors of the odd number $m \geq 3$.)

Remark 3.26. Inequality (1.8) is not generally true. Indeed, for $n = 39270$, $n = 82110$, or $n = 2\cdot3\cdot5\cdot7\cdot17\cdot23\cdot M$, where M is a Mersenne prime, greater or equal than 31, then (1.8) is not true. This has been communicated to the author by Professor L. Tóth. Prof. Kovács Lehel István found recently the counterexamples: 53130, 71610, 78540, 106260, 108570, 117810, 122430, 143220, 157080, 159390, 164010, 164220, 212520, 214830, 217140, 235620, 244860, 246330, 247170, 286440, 293370, 314160, 318780, 325710, 328440, 353430 and 367290.

Now by using a method of L. Alaoglu and P. Erdős [1], we will prove that:

Theorem 3.27. *For any $\delta > 0$, the inequality*

$$\varphi(\psi(n)) < \delta \cdot n \tag{3.40}$$

is valid, excepting perhaps $n \in S$, where S has asymptotic density zero.

Proof. We prove first that for any given prime p, the set of n such that $p|\psi(n)$, has density 1. This is similar to the proof given in [1].

On the other hand, since $\sum_{n\leq x}\psi(n)\approx\frac{15}{2\pi^2}\cdot x^2$ as $x\to\infty$ (see e.g. [16]), we can say that excepting at most a number of $\epsilon\cdot x$ integers $n<x$, one has $\psi(n)<c(\epsilon)\cdot n$, where $c(\epsilon)>0$.

Let now p be a prime such that

$$\prod_{q\leq p}\left(1-\frac{1}{q}\right)<\frac{\delta}{c(\epsilon)}$$

(this is possible, since $\prod_{q\leq p}\left(1-\frac{1}{q}\right)\to 0$ as $p\to\infty$).

Then, if x is large, then for all $n<x$, excepting perhaps a number of $\eta\cdot x+\epsilon\cdot x$ integers one has $\psi(n)<c(\epsilon)$. n and $\psi(n)\equiv 0(\operatorname{mod} q)$ for any $q\leq p,\ (\eta>0)$.

But for these exceptions one has $\varphi(\psi(n))<\delta\cdot n$, and this finishes the proof; $\eta,\epsilon>0$ being arbitrary. □

Remark 3.28. It can be proved similarly that

$$\psi(\varphi(n))>\delta\cdot n, \tag{3.41}$$

excepting perhaps a set of density zero.

Theorem 3.27 implies that $\lim\inf\limits_{n\to\infty}\frac{\varphi(\psi(n))}{n}=0$, and so, one has $\lim\sup\limits_{n\to\infty}\frac{\psi(\varphi(n))}{n}=+\infty$. For other proof of these results, see [16]. We cannot determine the following values: $\lim\inf\limits_{n\to\infty}\frac{\psi(\varphi(n))}{n}=$?, $\lim\sup\limits_{n\to\infty}\frac{\varphi(\psi(n))}{n}=$?

However, we can prove that:

Theorem 3.29.

$$\lim\inf_{n\to\infty}\frac{\psi(\varphi(n))}{n}\leq\inf\left\{\frac{\psi(\varphi(k))}{k}: k \textit{ is a multiple of } 4\right\}<\frac{1}{2}. \tag{3.42}$$

Proof. Let k be a multiple of 4, and $p>\frac{k}{2}$. Then

$$\varphi\left(\frac{1}{2}kp\right)=\varphi\left(\frac{k}{2}\right)\varphi(p)=2\varphi(\frac{k}{2})\cdot\frac{p-1}{2}=\varphi(k)\cdot\frac{p-1}{2},$$

since $2|\frac{k}{2}$. Now by $\psi(ab)\leq\psi(a)\psi(b)$ one can write

$$\psi\left(\varphi\left(\frac{1}{2}kp\right)\right)\leq\psi(\varphi(k))\psi\left(\frac{p-1}{2}\right).$$

Since $\psi\left(\frac{p-1}{2}\right)\leq\sigma\left(\frac{p-1}{2}\right)$, and by the known result of Makowski and Schinzel: $\lim\inf\frac{\sigma\left(\frac{p-1}{2}\right)}{\frac{p-1}{2}}=1$, from the above one can write:

$$\lim\inf_{p\to\infty}\frac{\psi\left(\varphi\left(\frac{1}{2}kp\right)\right)}{\frac{1}{2}kp}\leq\frac{\psi(\varphi(k))}{k}\cdot\lim\inf_{p\to\infty}\frac{\psi\left(\frac{p-1}{2}\right)}{\frac{p-1}{2}}\leq\frac{\psi(\varphi(k))}{k},$$

and now relation (3.42) follows, by taking $\inf$ after k.

Since

$$2^{32}-1=F_0\cdot F_1\cdot F_2\cdot F_3\cdot F_4,$$

where $F_k=2^{2^k}+1$, and all F_i $(0\leq i\leq 4)$ are primes, it follows, that

$$\varphi(2^{32}-1)=2^1\cdot 2^2\cdot 2^4\cdot 2^8\cdot 2^{16}=2^{31}.$$

Thus $\varphi(4(2^{32}-1)) = 2^{32}$, by $\varphi(4) = 2$. Since $\psi(2^{32}) = 2^{31}\cdot 3$, by letting in (3.42) $k = 4\cdot(2^{32}-1)$, we get the $\inf \le \frac{2^{31}\cdot 3}{4\cdot(2^{32}-1)} < \frac{1}{2\cdot(\frac{4}{3}-\theta)}$, where $\theta > \frac{1}{3\cdot 2^{30}}$. In any case we get in (3.42) that $\liminf < \frac{1}{2}$, and fact a value slightly greater than $\frac{1}{2\cdot\frac{4}{3}} = \frac{3}{8}$. $\square$

In [16] it is asked the value of $\liminf \frac{\psi(\sigma(n))}{n} \le 1$. We now prove that this value is 1:

Theorem 3.30.

$$\liminf \frac{\psi(\sigma(n))}{n} = 1. \tag{3.43}$$

Proof. Since $\frac{\psi(\sigma(n))}{n} \ge \frac{\sigma(n)}{n} \ge 1$, clearly this limit is ≥ 1. By the above inequality, the result follows. However, we give here a new proof of this fact. We remark that, since $\varphi(N) \le \psi(N) \le \sigma(N)$, and by the known result

$$\lim_{p\to\infty} \frac{\varphi(N(a,p))}{N(a,p)} = \lim_{p\to\infty} \frac{\sigma(N(a,p))}{N(a,p)} = 1,$$

where $N(a,p) = \frac{a^p-1}{p-1},\ (a>1, p \text{ prime})$ we easily get

$$\lim_{p\to\infty} \frac{\varphi(N(a,p))}{N(a,p)} = 1. \tag{3.44}$$

Now let $a = q$ an arbitrary prime in (3.44). We remark that $N(q,p) = \frac{q^p-1}{q-1} = \sigma(q^{p-1})$. Now, by

$$\frac{\sigma(q^{p-1})}{q^{p-1}} = \frac{q^p-1}{(q-1)\cdot q^{p-1}} \to \frac{q}{q-1},$$

as $p\to\infty$, from (3.44) we can write:

$$\lim_{p\to\infty} \frac{\psi(\sigma(q^{p-1}))}{q^{p-1}} = \frac{q}{q-1} < 1+\epsilon, \tag{3.45}$$

for $q \ge q(\epsilon)$, $\epsilon > 0$. Now by (3.45), (3.43) follows. $\square$

Remark 3.31. In [16] it is proved, by assuming the infinitude of Mersenne primes, that

$$\liminf_{n\to\infty} \frac{\psi(\psi(n))}{n} = \frac{3}{2}. \tag{3.46}$$

Can we prove (3.46) without any assumption?

We have conjectured in [16] that the following limit is true, but in the proof we have used the fact that there are infinitely many Mersenne primes. Now we prove this result without any assumptions:

Theorem 3.32. *We have*

$$\liminf \frac{\psi(\psi(n))}{n} = \frac{3}{2}. \tag{3.47}$$

Proof. Since $\psi(n) \ge \frac{3}{2}n$ for all even n, and $\psi(n) \ge n$ for all n, clearly $\psi(\psi(n)) \ge \frac{3}{2}\cdot n$ for all n, therefore it will be sufficient to find a sequence with limit $\frac{3}{2}$. By using deep theorems on primes in arithmetical progressions, it can be proved, as in Makowski-Schinzel [13] that

$$\limsup \frac{\varphi(a)}{a} = \liminf \frac{\sigma(a)}{a} = 1$$

as p tends to infinity, where $a = \frac{(p+1)}{2}$, and $p \equiv 1 \pmod 4$.

Since $\frac{(p+1)}{2}$ is odd, we get

$$\sigma(p+1) = \sigma\left(2 \cdot \frac{(p+1)}{2}\right) = 3 \cdot \sigma\left(\frac{(p+1)}{2}\right),$$

implying that $\liminf \frac{(\sigma(p+1))}{p} = \frac{3}{2}$. Since $\psi(n) \leq \sigma(n)$, we can write that $\liminf \frac{(\psi(p+1))}{p} \leq \frac{3}{2}$. By $\frac{(\psi(p+1))}{p} > \frac{3}{2}$, this yields $\liminf \frac{(\psi(p+1))}{p} = \frac{3}{2}$, completing the proof of the theorem. □

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