Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 2, Article 36, 2003

# A CRITERION FOR $p$-VALENTLY STARLIKENESS

MUHAMMET KAMALI
ATATURK UNIVERSITY,
FACULTY OF SCIENCE AND ARTS,
DEPARTMENT OF MATHEMATICS,
25240, ERZURUM-TURKEY.
mkamali@atauni.edu.tr

*Received 16 December, 2002; accepted 8 May, 2003*
*Communicated by A. Sofo*

ABSTRACT. It is the purpose of the present paper to obtain some sufficient conditions for $p$-valently starlikeness for a certain class of functions which are analytic in the open unit disk $E$.

*Key words and phrases:* $p-$valently starlikeness, Jack Lemma.

2000 *Mathematics Subject Classification.* 30C45, 31A05.

## 1. INTRODUCTION

Let $A(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in $E = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z) \in A(p)$ is said to be p-valently starlike if and only if

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0 \quad (z \in E).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are $p$-valently in $E$ (see, e.g., Goodman [1]).

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f(z)$ of the form (1.1) is said to be $\alpha-$convex in $E$ if it is regular,

$$\frac{f(z)}{z} f'(z) \neq 0,$$

ISSN (electronic): 1443-5756
© 2003 Victoria University. All rights reserved.
148-02

and

$$\text{Re}\left(\alpha\left(1+z\frac{f''(z)}{f'(z)}\right)+(1-\alpha)z\frac{f'(z)}{f(z)}\right)\geqslant 0 \tag{1.2}$$

for all $z$ in $E$. The set of all such functions is denoted by $\alpha-CV$, where $\alpha$ is a real number. Of course, if $\alpha=1$, then an $\alpha-$convex function is convex; and if $\alpha=0$, an $\alpha-$convex function is starlike. Thus the sets $\alpha-CV$ give a "continuous" passage from convex functions to starlike functions. Sakaguchi considers functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n,$$

where $p$ is a positive integer, and he imposes the condition

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right\}\geqslant 0 \tag{1.3}$$

for $z$ in $E$. He proved that if $k=-1$, there is only one function that satisfies (1.3), namely $f(z)\equiv z^p$. If $-1<k\leqslant 0$,then $f(z)$ is $p$-valent convex; and if $0<k$, then $f(z)$ is $p$-valent starlike. We can pass from (1.3) back to (1.2) if we divide by $1+k>0$ and set $\alpha=\frac{1}{1+k}$ [6]. We denote by $S(p,k)$ the subclass $A(p)$ consisting of functions which satisfy the condition (1.3).

Obradovic and Owa [7] have obtained a sufficient condition for starlikeness of $f(z)\in A(1)$ which satisfies a certain condition for the modulus of

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}},$$

we recall their result as:

**Theorem 1.1.** *If $f(z)\in A(1)$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<K\left|\frac{zf'(z)}{f(z)}\right|\quad (z\in E),$$

*where $K=1.2849...$, then $f(z)\in S(1)$.*

Nunokawa [4] improved Theorem 1.1 by proving

**Theorem 1.2.** *If $f(z)\in A(p)$, and if*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<\left|\frac{zf'(z)}{f(z)}\right|\frac{1}{p}\log(4e^{p-1})\quad (z\in E),$$

*then $f(z)\in S(p)$.*

## 2. PRELIMINARIES

In order to obtain our main result, we need the following lemma attributed to Jack [2] (given also by Miller and Mocanu [3]).

**Lemma 2.1.** *Let $w(z)$ be analytic in $E$ with $w(0)=0$. If $|w(z)|$ attains its maximum value in the circle $|z|=r<1$ at a point $z_0$, then we can write $z_0w'(z_0)=kw(z_0)$, where $k$ is a real number and $k\geq 1$.*

Making use of Lemma 2.1,we first prove

**Lemma 2.2.** *Let $q(z)$ be analytic in $E$ with $q(0) = p$ and suppose that*

$$\mathrm{Re}\left\{\frac{zq'(z)}{[q(z)]^2}\right\} < \frac{1}{p(\lambda+1)} \quad (z \in E, 0 \leqslant \lambda \leqslant 1), \tag{2.1}$$

then $\mathrm{Re}\{q(z)\} > 0$ in $E$.

*Proof.* Let us put

$$q(z) = p\left\{\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+w(z)}{1-w(z)} + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-w(z)}{1+w(z)}\right\},$$

where $0 \leqslant \lambda \leqslant 1$.

Then $w(z)$ is analytic in $E$ with $w(0) = 0$ and by an easy calculation, we have

$$1 + z\frac{q'(z)}{[q(z)]^2} = 1 + \frac{2}{p}\cdot\frac{(\lambda w^2(z) + 2w(z) + \lambda)zw'(z)}{(w^2(z) + 2\lambda w(z) + 1)^2}.$$

If we suppose that there exists a point $z_0 \in E$ such that $\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1$, then, from Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0), \quad (k \geqslant 1)$.

Putting $w(z_0) = e^{i\theta}$, we find that

$$\begin{aligned}
z_0\frac{q'(z_0)}{[q(z_0)]^2} &= \frac{2}{p}\cdot\frac{\lambda w^2(z_0)w'(z_0)z_0 + 2w(z_0)w'(z_0)z_0 + \lambda w'(z_0)z_0}{[w^2(z_0) + 2\lambda w(z_0) + 1]^2}\\
&= \frac{2k}{p}\cdot\frac{\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta}}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2}\\
&= \frac{2k}{p}\cdot\frac{\left(\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta}\right)}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2}\cdot\frac{\left(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1\right)^2}{(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1)^2}\\
&= \frac{k}{p}\cdot\frac{\lambda\cos 3\theta + (4\lambda^2+2)\cos 2\theta + (11\lambda + 4\lambda^3)\cos\theta + (8\lambda^2+2)}{4\left(\lambda+\cos\theta\right)^4}\\
&= \frac{k}{p}\cdot\frac{(1+\lambda\cos\theta)\left(\lambda+\cos\theta\right)^2}{(\lambda+\cos\theta)^4}\\
&= \frac{k}{p}\cdot\frac{1+\lambda\cos\theta}{(\lambda+\cos\theta)^2},
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{Re}\left\{z_0\frac{q'(z_0)}{[q(z_0)]^2}\right\} &= \frac{k}{p}\cdot\frac{1+\lambda\cos\theta}{(\lambda+\cos\theta)^2} = \frac{k}{p}\cdot\frac{\lambda^2+\lambda\cos\theta+1-\lambda^2}{(\lambda+\cos\theta)^2}\\
&= \frac{k}{p}\left\{\frac{\lambda}{(\lambda+\cos\theta)} + \frac{1-\lambda^2}{(\lambda+\cos\theta)^2}\right\}\\
&\geqslant \frac{1}{p}\left(\frac{1}{\lambda+1}\right).
\end{aligned}$$

This contradicts (2.1). Therefore, we have $|w(z)| < 1$ in $E$, and it follows that $\mathrm{Re}\{q(z)\} > 0$ in $E$. This completes our proof of Lemma 2.2. □

If we take $\lambda = 1$ in Lemma 2.2, then we have the following Lemma 2.3 by Nunokawa [5].

**Lemma 2.3.** *Let $q(z)$ be analytic in $E$ with $q(0) = p$ and suppose that*

$$\mathrm{Re}\left\{\frac{zq'(z)}{[q(z)]^2}\right\} < \frac{1}{2p} \quad (z \in E).$$

*Then* $\operatorname{Re}\{q(z)\} > 0$ *in* $E$.

## 3. A CRITERION FOR $p$-VALENTLY STARLIKENESS

**Theorem 3.1.** *Let* $f(z) \in A(p),\ f(z) \neq 0$, *in* $0 < |z| < 1$ *and suppose that*

$$\operatorname{Re}\left\{1 + z\frac{\left[1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)\right]'}{\left[1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)\right]^2}\right\} < 1 + \frac{1}{k+1}\left(\frac{1}{2p}\right) \quad (z \in E)\,. \tag{3.1}$$

*Then* $f(z) \in S(p,k)$.

*Proof.* Let us put

$$q(z) = \frac{1}{k+1}\left\{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right\} \quad (k > 0)\,.$$

Then, $q(z)$ is analytic in $E$ with $q(0) = p,\ q(z) \neq 0$ in $E$. We have

$$\frac{q'(z)}{q(z)} = \frac{\left(z\frac{f''(z)}{f'(z)}\right)' + \left(kz\frac{f'(z)}{f(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} = \frac{\frac{f''(z)}{f'(z)} + z\left(\frac{f''(z)}{f'(z)}\right)' + k\frac{f'(z)}{f(z)} + kz\left(\frac{f'(z)}{f(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}}.$$

Then, we obtain

$$\begin{aligned} z\frac{q'(z)}{q(z)} &= \frac{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)} - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} + z\frac{kz\left(\frac{f'(z)}{f(z)}\right)' + z\left(\frac{f''(z)}{f'(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} \\ &= 1 + \frac{z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' + k\left(\frac{f'(z)}{f(z)}\right)'\right] - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}}, \end{aligned}$$

or

$$\begin{aligned} &(k+1)q(z) + z\frac{q'(z)}{q(z)} \\ &\quad = 1 + \frac{z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' + k\left(\frac{f'(z)}{f(z)}\right)'\right] - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} + (k+1)q(z) \\ &\quad = 1 + \frac{z^2\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)' + 2z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right) + z^2\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)^2}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} \\ &\quad = 1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right) + z\frac{z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)' + \left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)}{\left(1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right)}. \end{aligned}$$

Thus,

$$1+\frac{1}{k+1}z\frac{q'(z)}{[q(z)]^2}=1+z\frac{z\left(\frac{f''(z)}{f'(z)}+k\frac{f'(z)}{f(z)}\right)'+\left(\frac{f''(z)}{f'(z)}+k\frac{f'(z)}{f(z)}\right)}{\left(1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right)^2}$$
$$=1+z\frac{\left[1+z\left(\frac{f''(z)}{f'(z)}+k\frac{f'(z)}{f(z)}\right)\right]'}{\left(1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right)^2}.$$

From Lemma 2.3 and (3.1), we thus find that

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right\}\geqslant 0 \quad (z\in E, k>0).$$

This completes our proof of Theorem 3.1. □

If we take $\alpha=0$, after writing $\frac{1}{k+1}=\alpha$ in (3.1), then we obtain M. Nunokawa's theorem as follows.

**Theorem 3.2.** *Let* $f(z)\in A(p)$, $f(z)\neq 0$, *in* $0<|z|<1$ *and suppose that*

$$\operatorname{Re}\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\}<1+\frac{1}{2p},\quad z\in E.$$

*Then* $f(z)\in S(p)$.

## REFERENCES

[1] A.W. GOODMAN, On the Schwarz-Christoffel transformation and $p$-valent functions, *Trans. Amer. Math. Soc.*, **68** (1950), 204–223.

[2] I.S. JACK, Functions starlike and convex of order $\alpha$, *J. London Math. Soc.*, **2**(3) (1971), 469–474.

[3] S.S. MILLER AND P.T. MOCANU, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (1978), 289–305.

[4] M. NUNOKAWA, On certain multivalent functions, *Math. Japon.*, **36** (1991), 67–70.

[5] M. NUNOKAWA, A certain class of starlike functions, in *Current Topics in Analytic Function Theory*, H.M. Srivastava and S. Owa (Eds.), Singapore, New Jersey, London, Hong Kong, 1992, p. 206–211.

[6] A.W. GOODMAN, *Univalent Functions*, Volume I, Florida, 1983, p.142–143.

[7] M. OBRADOVIC AND S. OWA, A criterion for starlikeness, *Math. Nachr.*, **140** (1989), 97–102.