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A FILIPPOV TYPE EXISTENCE THEOREM FOR A CLASS OF SECOND-ORDER DIFFERENTIAL INCLUSIONS

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ABSTRACT. We prove a Filippov-Gronwall type inequality for solutions of a nonconvex second-order differential inclusion of Sturm-Liouville type.

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1. INTRODUCTION

In this paper we study second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e. \ ([0,T]), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.1}$$

where $I = [0,T]$, $F : I \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, x_1 \in X$ and $p(\cdot) : [0,T] \to (0,\infty)$ is continuous.

In some recent papers ([3, 6]) several existence results for problem (1.1) were obtained using fixed point techniques. Even if we deal with an initial value problem instead of a boundary value problem, the differential inclusion (1.1) may be regarded as an extension to the set-valued framework of the classical Sturm-Liouville differential equation.

The aim of this paper is to show that Filippov's ideas ([4]) can be suitably adapted in order to prove the existence of solutions to problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem [4], well known in the literature as the Filippov-Gronwall inequality, consists in proving the existence of a solution satisfying some inequalities involving a given quasi trajectory.

Such an approach allows us to avoid additional hypotheses on the Lipschitz constant of the set-valued map that appear in the fixed point approaches ([3, 6]). The proof of our results follows the general ideas in [5], where a similar result is obtained for solutions of semilinear differential inclusions.

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The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main results.

2. PRELIMINARIES

Let us denote by I the interval $[0,T]$, $T>0$ and let X be a real separable Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot,\cdot)$. With B we denote the closed unit ball in X.

Consider $F : I \times X \to \mathcal{P}(X)$ a set-valued map, $x_0, x_1 \in X$ and $p(\cdot) : I \to (0,\infty)$ a continuous mapping that have defined the Cauchy problem (1.1).

A continuous mapping $x(\cdot) \in C(I,X)$ is called a solution of problem (1.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I,X)$ such that:

$$f(t) \in F(t,x(t)) \quad a.e.\,(I), \tag{2.1}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)}ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u)duds, \quad \forall t \in I. \tag{2.2}$$

This definition of the solution is justified by the fact that if $f(\cdot) \in L^1(I,X)$ satisfies (2.1), then from the equality $(p(t)x'(t))' = f(t)$ a.e. (I), integrating by parts and applying the Leibnitz-Newton formula for absolutely continuous functions twice, we obtain first

$$x'(t) = \frac{p(0)}{p(t)}x_1 + \frac{1}{p(t)}\int_0^t f(u)du, \quad t \in I \tag{2.3}$$

and afterwards (2.2).

Note that, if we denote $S(t,u) := \int_u^t \frac{1}{p(s)}$, $t \in I$, then (2.2) may be rewritten as

$$x(t) = x_0 + p(0)x_1 S(t,0) + \int_0^t S(t,u)f(u)du \quad \forall t \in I. \tag{2.4}$$

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notations for the solution sets of (1.1):

$$\mathcal{S}(x_0,x_1) = \{(x(\cdot), f(\cdot)); \quad (x(\cdot), f(\cdot)) \text{ is a trajectory-selection pair of (1.1)}\}. \tag{2.5}$$

In what follows $y_0, y_1 \in X$, $g(\cdot) \in L^1(I,X)$ and $y(\cdot)$ is a solution of the Cauchy problem

$$(p(t)y'(t))' = g(t) \quad y(0) = y_0, \quad y'(0) = y_1. \tag{2.6}$$

Hypothesis 2.1.

i) $F(\cdot,\cdot) : I \times X \to \mathcal{P}(X)$ *has nonempty closed values and for every* $x \in X$, $F(\cdot,x)$ *is measurable.*

ii) *There exist* $\beta > 0$ *and* $L(\cdot) \in L^1(I,(0,\infty))$ *such that for almost all* $t \in I$, $F(t,\cdot)$ *is* $L(t)$*-Lipschitz on* $y(t)+\beta B$ *in the sense that*

$$d_H(F(t,x_1),F(t,x_2)) \le L(t)|x_1 - x_2| \quad \forall x_1, x_2 \in y(t) + \beta B,$$

where $d_H(A,C)$ *is the Pompeiu-Hausdorff distance between* $A, C \subset X$

$$d_H(A,C) = \max\{d^*(A,C), d^*(C,A)\}, \quad d^*(A,C) = \sup\{d(a,C); a \in A\}.$$

iii) *The function* $t \to \gamma(t) := d(g(t), F(t,y(t)))$ *is integrable on* I.

Set $m(t) = e^{MT\int_0^t L(u)du}$, $t \in I$ and $M := \sup_{t\in I} \frac{1}{p(t)}$. Note that $|S(t,u)| \leq M(t-u) \leq Mt$ $\forall t, u \in I$, $u \leq t$.

On $C(I,X) \times L^1(I,X)$ we consider the following norm

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\, (x,f) \in C(I,X) \times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t\in I} |x(t)|$, $x \in C(I,X)$ and $|f|_1 = \int_0^T |f(t)|dt$, $f \in L^1(I,X)$.

The technical results summarized in the next lemma are well known in the theory of set-valued maps. For their proofs we refer, for example, to [5].

Lemma 2.2 ([5]). *Let X be a separable Banach space, $H : I \to \mathcal{P}(X)$ a measurable set-valued map with nonempty closed values and $g, h : I \to X, L : I \to (0,\infty)$ measurable functions. Then one has:*

i) *The function $t \to d(h(t), H(t))$ is measurable.*

ii) *If $H(t)\cap(g(t)+L(t)B) \neq \emptyset$ $a.e.$ (I) then the set-valued map $t \to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.*

Moreover, if Hypothesis 2.1 is satisfied and $x(\cdot) \in C(I,X)$ with $|x - y|_C \leq \beta$, then the set-valued map $t \to F(t, x(t))$ is measurable.

3. THE MAIN RESULTS

We are ready now to present a version of the Filippov theorem for the Cauchy problem (1.1).

Theorem 3.1. *Consider $\delta \geq 0$, assume that Hypothesis 2.1 is satisfied and set*

$$\eta(t) = m(t)(\delta + MT\int_0^t \gamma(s)ds).$$

If $\eta(T) \leq \beta$, then for any $x_0, x_1 \in X$ with

$$(|x_0 - y_0| + MTp(0)|x_1 - y_1|) \leq \delta$$

and any $\varepsilon > 0$ there exists $(x(\cdot), f(\cdot)) \in \mathcal{S}(x_0, x_1)$ such that

$$|x(t) - y(t)| \leq \eta(t) + \varepsilon MTtm(t) \quad \forall t \in I,$$

$$|f(t) - g(t)| \leq L(t)(\eta(t) + \varepsilon MTtm(t)) + \gamma(t) + \varepsilon \quad a.e.\ (I).$$

Proof. Let $\varepsilon > 0$ such that $\eta(T) + \varepsilon MT^2 m(T) < \beta$ and set

$$\chi(t) = \delta + MT\int_0^t \gamma(s)ds + \varepsilon MTt,$$

$x_0(t) \equiv y(t)$, $f_0(t) \equiv g(t)$, $t \in I$.

We claim that it is sufficient to construct the sequences $x_n(\cdot) \in C(I,X)$, $f_n(\cdot) \in L^1(I,X)$, $n \geq 1$ with the following properties

$$x_n(t) = x_0 + p(0)S(t,0)x_1 + \int_0^t S(t,s)f_n(s)ds, \quad \forall t \in I, \tag{3.1}$$

$$|x_1(t) - x_0(t)| \leq \chi(t) \quad \forall t \in I, \tag{3.2}$$

$$|f_1(t) - f_0(t)| \leq \gamma(t) + \varepsilon \quad a.e.\ (I), \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \geq 1, \tag{3.4}$$

$$|f_{n+1}(t)-f_n(t)|\leq L(t)|x_n(t)-x_{n-1}(t)| \quad a.e.\,(I), n\geq 1. \tag{3.5}$$

Indeed, from (3.1), (3.2) and (3.5) we have for almost all $t\in I$

$$\begin{aligned}|x_{n+1}(t)-x_n(t)| &\leq \int_0^t |S(t,t_1)|\cdot|f_{n+1}(t_1)-f_n(t_1)|dt_1\\ &\leq MT\int_0^t L(t_1)|x_n(t_1)-x_{n-1}(t_1)|dt_1\\ &\leq MT\int_0^t L(t_1)\int_0^{t_1}|S(t_1,t_2)|,\end{aligned}$$

$$\begin{aligned}|f_n(t_2)-f_{n-1}(t_2)|dt_2 &\leq (MT)^2\int_0^t L(t_1)\int_0^{t_1} L(t_2)|x_{n-1}(t_2)-x_{n-2}(t_2)|dt_2dt_1\\ &\leq (MT)^n\int_0^t L(t_1)\int_0^{t_1}L(t_2)\cdots\int_0^{t_{n-1}}L(t_n)|x_1(t_n)-y(t_n)|dt_n\dots dt_1\\ &\leq \chi(t)(MT)^n\int_0^t L(t_1)\int_0^{t_1}L(t_2)\cdots\int_0^{t_{n-1}}L(t_n)dt_n\dots dt_1\\ &=\chi(t)\frac{(MT\int_0^t L(s)ds)^n}{n!}.\end{aligned}$$

Therefore $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I,X)$. Thus, from (3.5) for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Moreover, from (3.2) and the last inequality we have

$$\begin{aligned}|x_n(t)-y(t)| &\leq |x_1(t)-y(t)|+\sum_{i=2}^{n-1}|x_{i+1}(t)-x_i(t)|\\ &\leq \chi(t)\left[1+MT\int_0^t L(s)ds+\frac{(MT\int_0^t L(s)ds)^2}{2!}+\cdots\right]\\ &\leq \chi(t)e^{MT\int_0^t L(s)ds}\\ &=\eta(t)+\varepsilon MTtm(t)\end{aligned} \tag{3.6}$$

and taking into account the choice of ε, we get

$$|x_n(\cdot)-y(\cdot)|_C\leq\beta, \quad \forall n\geq 0. \tag{3.7}$$

On the other hand, from (3.3), (3.5) and (3.6) we obtain for almost all $t\in I$

$$\begin{aligned}|f_n(t)-g(t)| &\leq \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-g(t)|\\ &\leq L(t)\sum_{i=1}^{n-2}|x_i(t)-x_{i-1}(t)|+\gamma(t)+\varepsilon\\ &\leq L(t)(\eta(t)+\varepsilon tm(t))+\gamma(t)+\varepsilon.\end{aligned} \tag{3.8}$$

Let $x(\cdot)\in C(I,X)$ be the limit of the Cauchy sequence $x_n(\cdot)$. From (3.8) the sequence $f_n(\cdot)$ is integrably bounded and we have already proved that for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Take $f(\cdot)\in L^1(I,X)$ with $f(t)=\lim_{n\to\infty}f_n(t)$.

Using Hypothesis 2.1 iii) we have that for almost all $t \in I$, the set

$$Q(t) = \{(x, v); v \in F(t, x), |x - y(t)| \leq \beta\}$$

is closed. In addition, (3.4) and (3.7) imply that for $n \geq 1$ and $t \in I$, $(x_{n-1}(t), f_n(t)) \in Q(t)$. So, passing to the limit we deduce that (2.1) holds true for almost all $t \in I$.

Moreover, passing to the limit in (3.1) and using Lebesque's dominated convergence theorem we get (2.4). Finally, passing to the limit in (3.6) and (3.8) we obtained the desired estimations.

It remains to construct the sequences $x_n(\cdot), f_n(\cdot)$ with the properties in (3.1) – (3.5). The construction will be done by induction.

We apply, first, Lemma 2.2 and we have that the set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{g(t) + (\gamma(t) + \varepsilon)B\} \neq \emptyset \quad a.e. \ (I).$$

From Lemma 2.2 we find $f_1(\cdot)$ a measurable selection of the set-valued map

$$H_1(t) := F(t, y(t)) \cap \{g(t) + (\gamma(t) + \varepsilon)B\}.$$

Obviously, $f_1(\cdot)$ satisfy (3.3). Define $x_1(\cdot)$ as in (3.1) with $n = 1$. Therefore, we have

$$|x_1(t) - y(t)| \leq |x_0 - y_0| + |p(0)S(t, 0)(x_1 - y_1)| + \left| \int_0^t S(t, s)(f_1(s) - g(s))ds \right|$$
$$\leq \delta + M \int_0^t (\gamma(s) + \varepsilon)ds \leq \eta(t) + MT\varepsilon t \leq \beta.$$

Assume that for some $N \geq 1$ we already constructed $x_n(\cdot) \in C(I, X)$ and $f_n(\cdot) \in L^1(I, X), n = 1, 2, \ldots, N$ satisfying (3.1) – (3.5). We define the set-valued map

$$H_{N+1}(t) := F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|B\}, \quad t \in I.$$

From Lemma 2.2 the set-valued map $t \to F(t, x_N(t))$ is measurable and from the lipschitzianity of $F(t, \cdot)$ we have that for almost all $t \in I$, $H_{N+1}(t) \neq \emptyset$. We apply Lemma 2.2 and find a measurable selection $f_{N+1}(\cdot)$ of $F(\cdot, x_N(\cdot))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e. \ (I)$$

We define $x_{N+1}(\cdot)$ as in (3.1) with $n = N + 1$ and the proof is complete. □

Remark 1. As one can see from the proof of Theorem 3.1 the function $f(\cdot)$ is obtained to be integrable and so the function $t \to \int_0^t f(s)ds$ is at most absolutely continuous. Taking into account (2.3), if we assume that $p(\cdot)$ is absolutely continuous we find that $x(\cdot)$, the solution of (1.1), belongs to the space of differentiable functions whose first derivative $x'(\cdot)$ is absolutely continuous.

The next corollary of Theorem 3.1 shows the Lipschitz dependence of the solutions with respect to the initial conditions.

Corollary 3.2. *Let (y, g) be a trajectory-selection of (1.1) and assume that Hypothesis 2.1 is satisfied. Then there exists a $K > 0$ such that for any $\eta = (\eta_1, \eta_2)$ in a neighborhood of $(y(0), y'(0))$ we have*

$$d_{C \times L}((y, g), \mathcal{S}(\eta_1, \eta_2)) \leq K(|\eta_1 - y(0)| + |\eta_2 - y'(0)|).$$

Proof. Take $0 < \varepsilon < 1$. We apply Theorem 3.1 and deduce the existence of $\delta > 0$ such that for any $\eta = (\eta_1, \eta_2) \in B((y(0), y'(0)), \delta)$ there exists a trajectory-selection $(x_\varepsilon, f_\varepsilon)$ of (1.1) with $x_\varepsilon(0) = \eta_1$ and $x'_\varepsilon(0) = \eta_2$ such that

$$|x_\varepsilon - y|_C \leq m(T)(|\eta_1 - y(0)| + p(0)MT|\eta_2 - y'(0)|) + \varepsilon MT^2 m(T)$$

and

$$|f_\varepsilon - g|_1 \le m(T)(|\eta_1 - y(0)| + MT|\eta_2 - y'(0)|) + \varepsilon(MT^2 m(T) + 1).$$

Since $\varepsilon > 0$ is arbitrary the proof is complete. □

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