MEANS AND GENERALIZED MEANS

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Received: 08 May, 2007

Accepted: 28 May, 2007

Communicated by: P.S. Bullen

2000 AMS Sub. Class.: 26E60.

Key words: Gini means, Power means, Generalized means, Complementary means, Double sequences.

Abstract: In this paper, the Gaussian product of generalized means (or reflexive functions) is considered and an invariance principle for generalized means is proved.

Means and Generalized Means
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vol. 8, iss. 2, art. 45, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Means

A general abstract definition of means can be given in the following way. Let D be a set in $\mathbb{R}_+^2$ and M be a real function defined on D.

Definition 1.1. *We call the function M a mean on D if it has the property*

$$\min(a,b) \le M(a,b) \le \max(a,b), \qquad \forall\,(a,b) \in D.$$

In the special case $D = J^2$, where $J \subset \mathbb{R}_+$ is an interval, the function M is called mean on J.

Remark 1.1. Each mean is *reflexive* on its domain of definition D, that is

$$M(a,a) = a, \qquad \forall\,(a,a) \in D.$$

A function M (not necessarily a mean) can have some special properties.

Definition 1.2.

i) *The function M is symmetric on D if $(a,b) \in D$ implies $(b,a) \in D$ and*

$$M(a,b) = M(b,a), \qquad \forall\,(a,b) \in D.$$

ii) *The function M is homogeneous (of degree one) on D if there exists a neighborhood V of 1 such that $t \in V$ and $(a,b) \in D$ implies $(ta,tb) \in D$ and*

$$M(ta,tb) = tM(a,b).$$

iii) *The function M is strict at the left (respectively strict at the right) on D if for $(a,b) \in D$*

$$M(a,b) = a \text{ (respectively } M(a,b) = b\text{), implies } a = b.$$

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iv) The function M is strict if it is strict at the left and strict at the right.

The operations with means are considered as operations with functions. For instance, given the means M and N define $M \cdot N$ by

$$(M \cdot N)(a,b) = M(a,b) \cdot N(a,b), \quad \forall a,b \in D.$$

We shall refer to the following means on $\mathbb{R}_+$ (see [2]):

– the *weighted Gini mean* defined by

$$\mathcal{B}_{r,s;\lambda}(a,b) = \left[\frac{\lambda \cdot a^r + (1-\lambda) \cdot b^r}{\lambda \cdot a^s + (1-\lambda) \cdot b^s}\right]^{\frac{1}{r-s}}, \ r \neq s,$$

with $\lambda \in [0,1]$ fixed;

– the special case of the *weighted power mean* $\mathcal{B}_{r,0;\lambda} = \mathcal{P}_{r;\lambda}, r \neq 0$;

– the *weighted arithmetic mean* $\mathcal{A}_\lambda = \mathcal{P}_{1;\lambda}$;

– the *weighted geometric mean*

$$\mathcal{G}_\lambda(a,b) = a^\lambda b^{1-\lambda};$$

– the corresponding symmetric means, obtained for $\lambda = 1/2$ and denoted by $\mathcal{B}_{r,s}, \mathcal{P}_r, \mathcal{A}$ respectively $\mathcal{G}$;

– for $\lambda = 0$ or $\lambda = 1$, we have

$$\mathcal{B}_{r,s;0} = \Pi_2 \text{ respectively } \mathcal{B}_{r,s;1} = \Pi_1 , \quad \forall r,s \in \mathbb{R} ,$$

where we denoted by Π_1 and Π_2 the first respectively the second projections defined by

$$\Pi_1(a,b) = a, \ \Pi_2(a,b) = b, \quad \forall a,b \geq 0.$$

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2. Generalized Means

Let D be a set in $\mathbb{R}_+^2$ and M be a real function defined on D. In [6] the following was used:

Definition 2.1. *The function M is called a generalized mean on D if it has the property*

$$M(a,a) = a, \qquad \forall (a,a) \in D.$$

Remark 2.1. Each mean is reflexive, thus it is a generalized mean. Conversely, each generalized mean on D is a mean on $D \cap \Delta$, where

$$\Delta = \{(a,a)\,; a \geq 0\}\,.$$

The question is if the set $D \cap \Delta$ can be extended. The answer is generally negative. Take for instance the generalized mean $\mathcal{B}_{r,s;\lambda}$ for $\lambda \notin [0,1]$. Even though it is defined on a larger set like

$$\left(\frac{\lambda}{\lambda-1}\right)^{1/s} \leq \frac{b}{a} \leq \left(\frac{\lambda}{\lambda-1}\right)^{1/r}, \qquad \text{for } \lambda > 1, r > s > 0,$$

it is a mean only on Δ. However, the above question may have also a positive answer. For example, in [6], the following was proved.

Theorem 2.2. *If M is a differentiable homogeneous generalized mean on $\mathbb{R}_+^2$ such that*

$$0 < M_b(1,1) < 1,$$

then there exists the constants $T' < 1 < T"$ such that M is a mean on

$$D = \{(a,b) \in \mathbb{R}_+^2; T'a \leq b \leq T"a\}.$$

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We can strengthen the previous result by dropping the hypothesis of homogeneity for the generalized mean M.

Theorem 2.3. *If M is a differentiable generalized mean on the open set D such that*

$$0 < M_b(a,a) < 1, \quad \forall (a,a) \in D,$$

then for each $(a,a) \in D$ there exist the constants $T'_a < 1 < T_a$" such that

$$ta \leq M(a,ta) \leq a; \quad T'_a \leq t \leq 1$$

and

$$a \leq M(a,ta) \leq ta; \quad 1 \leq t \leq T''_a.$$

Proof. Let us consider the auxiliary functions defined by:

$$f(t) = M(a,ta) - a, g(t) = ta - M(a,ta),$$

in a neighborhood of 1 . Then there exist the numbers $T'_a < 1 < T''_a$ such that

$$f'(t) = aM_b(a,ta) \geq 0, \quad t \in \left(T'_a, T''_a\right)$$

and

$$g'(t) = a - aM_b(a,ta) \geq 0, \quad t \in \left(T'_a, T''_a\right).$$

As

$$f(1) = g(1) = 0,$$

the conclusions follow. □

Example 2.1. Let us take $M = \mathcal{A}_\lambda^2/\mathcal{G}$. As $M_b(1,1) = (3-4\lambda)/2$, the previous result is valid for M if $\lambda \in (0.25, 0.75)$. Looking at the set D on which M is a mean, for $a \leq b$ we have to verify the inequalities

$$a \leq \frac{[\lambda a + (1-\lambda) b]^2}{\sqrt{ab}} \leq b.$$

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Denoting $a/b = t^2 \in [0,1]$, we get the equivalent system

$$\begin{cases} \lambda^2 t^4 - t^3 + 2\lambda(1-\lambda)t^2 + (1-\lambda)^2 \geq 0, \\ \lambda^2 t^4 + 2\lambda(1-\lambda)t^2 - t + (1-\lambda)^2 \leq 0. \end{cases}$$

A similar system can be obtained for the case $a > b$. Solving these systems, we obtain a table with the interval $(T', T")$ for some values of λ :

λ	T'	T"
0.25	0.004...	1.
0.3	0.008...	1.671...
0.5	0.087...	11.444...
0.7	0.598...	113.832...
0.75	1.0	243.776...

For $\lambda \notin [0.25, 0.75]$, we get $T' = T" = 1$.

Remark 2.2. A similar result can be proved in the case

$$0 < M_a(b,b) < 1, \quad \forall (b,b) \in D.$$

If the partial derivatives do not belong to the interval $(0,1)$, the result can be false.

Example 2.2. For $M = \mathcal{B}_{r,s;\lambda}$, we have $M_b(a,a) = 1 - \lambda$. As we remarked, for $\lambda \notin [0,1]$ the generalized Gini mean is a mean only on Δ.

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issn: 1443-5756

3. Complementary Means

Let us now consider the following notion. Two means M and N are said to be *complementary* (with respect to $\mathcal{A}$) ([4]) if $M + N = 2 \cdot \mathcal{A}$. They are called *inverse* (with respect to $\mathcal{G}$) if $M \cdot N = \mathcal{G}^2$. In [5] a generalization was proposed, replacing $\mathcal{A}$ and $\mathcal{G}$ by an arbitrary mean P.

Given three functions M, N and P on D, their *composition* $P(M, N)$ can be defined on $D' \sqsubseteq D$ by

$$P(M, N)(a, b) = P(M(a, b), N(a, b)), \quad \forall (a, b) \in D',$$

if $(M(a, b), N(a, b)) \in D$, $\forall (a, b) \in D'$. If M, N and P are means on D then $D' = D$.

Definition 3.1. *A function N is called complementary to M with respect to P (or $P-$ complementary to M) if it verifies*

$$P(M, N) = P \text{ on } D'.$$

Remark 3.1. In the same circumstances, the function P is called $(M, N)-$ *invariant* (see [1]).

If M has a unique $P-$complementary N , denote it by $N = M^P$. We get

$$M^{\mathcal{A}} = 2\mathcal{A} - M \text{ and } M^{\mathcal{G}} = \mathcal{G}^2/M,$$

as in [4].

Remark 3.2. If P and M are means, the $P-$complementary of M is generally not a mean.

Example 3.1. It can be verified that

$$\mathcal{G}_\mu^{\mathcal{G}_\lambda} = \mathcal{G}_{\frac{\lambda(1-\mu)}{1-\lambda}},$$

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issn: 1443-5756

which is a mean if and only if $0 < \lambda < 1/(2-\mu)$.

For generalized means we get the following result.

Theorem 3.2. *If P and M are generalized means and P is strict at the left, then the $P-$ complementary of M is a generalized mean N.*

Proof. We have

$$P(M(a,a), N(a,a)) = P(a,a), \quad \forall (a,a) \in D,$$

thus

$$P(a, N(a,a)) = a, \quad \forall (a,a) \in D$$

and as P is strict at the left, we get $N(a,a) = a$, $\forall (a,a) \in D$. □

The result cannot be improved for means, thus we have only the following

Corollary 3.3. *If P and M are means and P is strict at the left, then the $P-$complementary of M is a generalized mean N.*

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4. Double Sequences

An important application of complementary means is in the search of Gaussian double sequences with known limit. The arithmetic-geometric process of Gauss can be generalized as follows. Let us consider two functions M and N defined on a set D and let $(a, b) \in D$ be an initial point.

Definition 4.1. *If the pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ can be defined by*

$$a_{n+1} = M(a_n, b_n) \quad \text{and} \quad b_{n+1} = N(a_n, b_n)$$

for each $n \geq 0$, where $a_0 = a$, $b_0 = b$, then it is called a Gaussian double sequence. The function M is compoundable in the sense of Gauss (or **G***-compoundable) with the function N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are defined and convergent to a common limit $M \otimes N(a, b)$ for each $(a, b) \in D$. In this case $M \otimes N$ is called the Gaussian compound function (or* **G***-compound function).*

Remark 4.1. If M and N are G-compoundable means, then $M \otimes N$ is also a mean called the **G**-*compound mean.*

The following general result was proved in [3].

Theorem 4.2. *If the means M and N are continuous and strict at the left on an interval J then M and N are G-compoundable on J.*

A similar result is valid for means which are strict at the right. In [5] the same result was proved assuming that one of the means M and N is continuous and strict.

In the case of means, the method of search of G-compound functions is based generally on the following *invariance principle*, proved in [1] .

Theorem 4.3. *Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P which is (M, N)-invariant.*

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In the same way, Gauss proved that the arithmetic-geometric G-compound mean can be represented by

$$\mathcal{A} \otimes \mathcal{G}(a,b) = \frac{\pi}{2} \cdot \left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right]^{-1}.$$

This example shows that the search of an invariant mean is very difficult even for simple means like $\mathcal{A}$ and $\mathcal{G}$. We prove the following generalization of the invariance principle.

Theorem 4.4. *Let P be a continuous generalized mean on D and M and N be two functions on D such that N is the $P-$ complementary of M. If the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by*

$$a_{n+1} = M(a_n, b_n) \text{ and } b_{n+1} = N(a_n, b_n), \qquad n \geq 0,$$

are convergent to a common limit L denoted as $M \otimes N(a_0, b_0)$, then this limit is

$$M \otimes N(a_0, b_0) = P(a_0, b_0).$$

Proof. As N is the $P-$ complementary of M, we have

$$P(M(a_n, b_n), N(a_n, b_n)) = P(a_n, b_n), \quad \forall n \geq 0,$$

thus

$$P(a_{n+1}, b_{n+1}) = P(a_n, b_n), \quad \forall n \geq 0.$$

But this also means that

$$P(a_0, b_0) = P(a_n, b_n), \quad \forall n \geq 0.$$

Finally, as P is a continuous generalized mean, passing to the limit we get

$$P(a_0, b_0) = P(L, L) = L,$$

which proves the result. □

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It is natural to study the following

Problem 4.1. If N is the $P-$complementary of M but M, N or P are not means, are the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ convergent?

The answer can be positive as it is shown in the following

Example 4.1. We have $\mathcal{G}_{5/8}^{\mathcal{G}_{4/5}} = \mathcal{G}_{3/2}$, where $\mathcal{G}_{3/2}$ is not a mean. Take $a_0 = 10^5, b_0 = 1$ and

$$a_{n+1} = \mathcal{G}_{5/8}(a_n, b_n), \quad b_{n+1} = \mathcal{G}_{3/2}(a_n, b_n), \quad n \geq 0.$$

Although some of the first terms take values outside the interval $[b_0, a_0]$ like

$$b_1 \approx 3.1 \cdot 10^7, \quad b_3 \approx 4.7 \cdot 10^6, \quad b_5 \approx 1.1 \cdot 10^6, \quad b_7 \approx 3.7 \cdot 10^5, \quad b_9 \approx 1.5 \cdot 10^5,$$

finally we get $a_{100} = 9999.9\ldots, b_{100} = 10000.1\ldots$, while $\mathcal{G}_{4/5}(a_0, b_0) = 10^4$.

But the answer to the above problem can be also negative.

Example 4.2. We have $\mathcal{G}_2^{\mathcal{G}_{-1}} = \mathcal{G}$, but taking $a_0 = 10, b_0 = 1$ and

$$a_{n+1} = \mathcal{G}_2(a_n, b_n) \text{ and } b_{n+1} = \mathcal{G}(a_n, b_n), \quad n \geq 0,$$

we get $a_3 = 10^9, b_3 = 4 \cdot 10^6$ and the sequences are divergent. In this case $\mathcal{G}_2$ and $\mathcal{G}_{-1}$ are not means.

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issn: 1443-5756

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