Journal of Inequalities in Pure and Applied Mathematics

CONTINUITY PROPERTIES OF CONVEX-TYPE SET-VALUED MAPS

KAZIMIERZ NIKODEM

Department of Mathematics,
University of Bielsko–Biała Willowa 2,
PL-43-309 Bielsko–Biała, Poland.
E-Mail: knik@ath.bielsko.pl

ISSN (electronic): 1443-5756
154-02

volume 4, issue 3, article 52, 2003.

Received 29 December, 2002; accepted 21 May, 2003.

Communicated by: Z. Páles

It is well known that convex functions defined on an infinite–dimensional space need not be continuous and midconvex (Jensen convex) functions, they may be discontinuous even if they are defined on an open interval in $\mathbb{R}$. However, their continuity follows from other regularity assumptions, such as continuity at a point, upper boundedness on a set with non–empty interior, measurability, lower semicontinuity, closedness of the epigraph, etc. (cf. e.g. [26], [12]). The aim of this note is to collect similar results for convex set–valued maps. Such maps arise naturally from, e.g., the constraints of convex optimization problems and play an important role in various questions of convex analysis and economic theory (cf. [4], [5], [13], [27], [28], [29] for more information). Conditions implying their continuity can be found, among others, in [3], [6], [7], [8], [9], [16], [17], [18], [19], [20], [22], [23], [24], [25], [27], [30], [31].

Let X and Y be topological vector spaces (real and Hausdorff in the whole paper), D be a convex subset of X and K be a convex cone in Y (i.e. $K + K \subset K$ and $tK \subset K$ for all $t \geq 0$). Denote by $n(Y)$, $b(Y)$, $c(Y)$ and $cc(Y)$ the families of all non–empty, non–empty bounded, non–empty compact and non–empty compact convex subsets of Y, respectively.

A set–valued map (s.v. map for short) $F : D \to n(Y)$ is said to be *K–convex* if

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + K \tag{1}$$

for all $x_1, x_2 \in D$ and $t \in [0, 1]$; F is called *K–midconvex* (or *K– Jensen convex*) if

$$\frac{F(x_1) + F(x_2)}{2} \subset F\left(\frac{x_1 + x_2}{2}\right) + K, \tag{2}$$

for all $x_1, x_2 \in D$. Equivalently, F is K–convex (K–midconvex) if its *epigraph*, i.e. the set

$$epi F = \{(x,y) \in D \times Y : y \in F(x) + K\},$$

is a convex (midconvex) subset of $X \times Y$.

Note that F is K–convex (K–midconvex) with $K = \{0\}$ iff its *graph*, i.e. the set

$$gr F = \{(x,y) \in D \times Y : y \in F(x)\},$$

is a convex (midconvex) subset of $X \times Y$.

If F is single–valued and Y is endowed with the relation $\leq_K$ of partial order defined by $x \leq_K y :\Longleftrightarrow y - x \in K$, then condition (1) reduces to the following one

$$F(tx_1 + (1-t)x_2) \leq_K tF(x_1) + (1-t)F(x_2).$$

In particular if $Y = \mathbb{R}$ and $K = [0, \infty)$, we obtain the standard definition of convex functions.

We say that a set–valued map $F : D \to n(Y)$ is *K–continuous* at a point $x_0 \in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + W + K \tag{3}$$

and

$$F(x) \subset F(x_0) + W + K \tag{4}$$

for every $x \in (x_0 + U) \cap D$. Only when condition (3) (condition (4)) is fulfilled, we say that F is *K–lower semicontinuous* (*K–upper semicontinuous*) at x_0. The

K–continuity in the case where $K = \{0\}$ means the continuity with respect to the Hausdorff topology on $n(Y)$. If K is a normal cone (i.e. if there exists a base $\mathcal{W}$ of neighbourhoods of zero in Y such that $W = (W - K) \cap (W + K)$ for every $W \in \mathcal{W}$) and F is a single–valued function, then K–continuity means continuity. Note also that in the case where F is a real–valued function and $K = [0, \infty)$ then conditions (3) and (4) define the classical upper and lower semicontinuity of F at x_0, respectively.

We start with the following result showing that for K–midconvex s.v. maps K–lower semicontinuity at a point implies K–continuity on the whole domain.

Theorem 1. *([17, Thm. 3.3]; cf. also [6]). Let X and Y be topological vector spaces, D be a convex open subset of X, and K be a convex cone in Y. Assume that $F : D \to b(Y)$ and $G : D \to n(Y)$ are s.v. maps such that $G(x) \subset F(x) + K$, for all $x \in D$. If F is K–midconvex and G is K–lower semicontinuous at a point of D, then F is K–continuous on D.*

As an immediate consequence of this theorem (under the same assumptions on X, Y, D and K) we get the following corollaries. Recall that a function $f : D \to Y$ is a *selection* of $F : D \to n(Y)$ if $f(x) \in F(x)$ for all $x \in D$.

Corollary 2. *If a s.v. map $F : D \to b(Y)$ is K–midconvex and K–lower semicontinuous at a point of D, then it is K–continuous on D.*

Corollary 3. *If a s.v. map $F : D \to b(Y)$ is K–midconvex and has a selection continuous at a point of D, then it is K–continuous on D.*

In the centre of many results giving conditions under which midconvex (or convex) functions are continuous there are two basic theorems. The first one is the theorem of Bernstein and Doetsch, stating that midconvex functions bounded above on a set with non-empty interior are continuous, and the second one is the theorem of Sierpiński, stating that measurable midconvex functions are continuous (cf. [26], [12]). The next two theorems are far–reaching generalizations of those results for K–midconvex s.v. maps.

We say that an s.v. map F is K*–upper bounded* on a set A if there exists a bounded set $B \subset Y$ such that $F(x) \cap (B - K) \neq \emptyset$, for all $x \in A$.

Theorem 4. *([17, Thm. 3.4]). Let X and Y be topological vector spaces, D – an open convex subset of X and K – a convex cone in Y. If an s.v. map $F : D \to b(Y)$ is K–midconvex and K–upper bounded on a subset of D with non–empty interior, then F is K–continuous on D.*

Remark 1. *In the case where $X = \mathbb{R}^n$, it is sufficient to assume that the set A is of positive Lebesgue measure. Indeed, if F is K–upper bounded on A, then, by the K–midconvexity, it is also K–upper bounded on the set $(A + A)/2$, which, by the classical Steinhaus theorem, has non-empty interior.*

Recall that a set–valued map $F : \mathbb{R}^n \supset D \to n(Y)$ is *Lebesgue measurable* if for every open set $W \subset Y$ the set

$$F^+(W) = \{t \in D : F(x) \subset W\}$$

is Lebesgue measurable.

Theorem 5. *([17, Thm. 3.8]; cf. also [30]). Let D be a convex open subset of $\mathbb{R}^n$, Y be a topological vector space, and K be a convex cone in Y. Assume that $F : D \to b(Y)$ and $G : D \to b(Y)$ are s.v. maps such that $G(x) \subset F(x) + K$, for all $x \in D$. If F is K–midconvex and G is Lebesgue measurable, then F is K–continuous on D.*

Under the same assumptions on D, Y and K we have the following corollaries.

Corollary 6. *If a s.v. map $F : D \to b(Y)$ is K–midconvex and Lebesgue measurable, then it is K–continuous on D.*

Corollary 7. *If a s.v. map $F : D \to b(Y)$ is K–midconvex and has a Lebesgue measurable selection, then it is K–continuous on D.*

The next result generalizes the well known result stating that convex functions defined on an open subset of a finite–dimensional space are continuous.

Theorem 8. ([17, *Thm. 3.7]; cf. also* [24]). *Let D be a convex open subset of $\mathbb{R}^n$, Y be a topological vector space, and K be a convex cone in Y. If a s.v. map $F : D \to b(Y)$ is K–convex, then it is K–continuous on D.*

Now we present a generalization of the classical closed graph theorem.

Theorem 9. ([18, *Thm. 1]). Let X be a Baire topological vector space, D be a convex open subset of X, Y be a locally convex topological vector space and K be a convex cone in Y. Assume that there exist compact sets $B_n \subset Y$, $n \in \mathbb{N}$, such that*

$$\bigcup_{n\in\mathbb{N}} (B_n - K) = Y. \tag{5}$$

If a s.v. map $F : D \to b(Y)$ is K–midconvex and its epigraph is closed in $D \times Y$, then it is K–continuous on D.

Remark 2. *The assumption (5) is trivially satisfied if Y is a locally compact space (and K is an arbitrary convex cone in Y). It is also fulfilled if there exists an order unit in Y, i.e. such an element $e \in Y$ that for every $y \in Y$ we can find an $n \in \mathbb{N}$ with $y \in ne - K$ (we put then $B_n = \{ne\}$). In particular, if $int\ K \neq \emptyset$, then every element of $int\ K$ is an order unit in Y. The above result extends the closed graph theorem proved by Ger [10] for midconvex operators and crosses with the closed graph theorems due to Borwein [6], Ricceri [25] and Robinson-Ursescu [27], [31] (cf. also [2]).*

The next result generalizes the known theorem stating that lower semicontinuous convex functions are continuous. Given a convex cone K in a topological vector space Y we denote by K^* the set of all continuous linear functionals on Y which are nonnegative on K, i.e.

$$K^* = \{y^* \in Y^* : y^*(y) \geq 0, \text{ for every } y \in K\}.$$

Theorem 10. ([19, Thm. 1]). *Let X be a Baire topological vector space, D – a convex open subset of X, Y – a locally convex topological vector space and K – a convex cone in Y. Moreover, assume that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that condition (5) holds. If a s.v. map $F : D \to cc(Y)$ is K–midconvex and for every $y^* \in K^*$ the functional $x \longmapsto f_{y^*}(x) = \inf\ y^*(F(x))$, $x \in D$, is lower semicontinuous on D, then F is K–continuous on D.*

It is easy to check that if a s.v. map $F : D \to b(Y)$ is K–upper semicontinuous at a point, then for every $y^* \in K^*$ the functional f_{y^*} defined above is lower semicontinuous at this point. Therefore, as a consequence of the above theorem, we get the following result.

Corollary 11. *Let X, D, Y and K be such as in Theorem 10. If a K–midconvex s.v. map $F : D \to cc(Y)$ is K–upper semicontinuous on D, then it is K–continuous on D.*

Now we will present the Mazur's criterion for continuity of K–midconvex s.v. maps. It is related to the following question posed by S. Mazur [15]: In a Banach space E there is given an additive functional f such that, for every continuous function $x : [0, 1] \to E$, the superposition $f \circ x$ is Lebesgue measurable. Is f continuous?

The answer to that question, in the affirmative, was given by I. Labuda and R.D. Mauldin [14]. R. Ger [11] showed that the same remains true in the case where f is a midconvex functional defined on an open convex subset D of E. More precisely, he proved that each midconvex functional $f : D \to E$ such that for every continuous function $x : [0, 1] \to D$, the superposition $f \circ x$ admits a Lebesgue measurable majorant, is continuous. The next theorem is a set-valued generalization of this result.

Theorem 12. ([20, *Thm. 1*]). *Let E be a real Banach space, D – an open convex subset of E, Y – a locally convex topological vector space and K – a convex cone in Y. Moreover, assume that there exist bounded sets $B_n \subset Y$, $n \in \mathbb{N}$, such that condition (5) holds. If a set–valued map $F : D \to cc(Y)$ is K–midconvex and for every continuous function $x : [0, 1] \to D$ there exists a Lebesgue measurable set–valued map $G : [0, 1] \to c(Y)$ such that*

$$G(t) \subset F(x(t)) + K \ , \ t \in [0, 1],$$

then F is K–continuous on D.

As an immediate consequence of the above theorem (under the same assumptions on E, D, Y and K) we obtain the following corollaries.

Corollary 13. *If a set–valued map $F : D \to cc(Y)$ is K–midconvex and for every continuous function $x : [0,1] \to D$ the superposition $F \circ x$ is Lebesgue measurable, then F is K–continuous on D.*

Corollary 14. *If a set–valued map $F : D \to cc(Y)$ is K–midconvex and for every continuous function $x : [0,1] \to D$ the superposition $F \circ x$ has a Lebesgue measurable selection, then F is K–continuous on D.*

Now assume that $\lambda : D^2 \to (0,1)$ is a fixed function. We say that a set-valued map $F : D \to n(Y)$ is (K,λ)-*convex* if

$$\lambda(x,y)F(x) + (1-\lambda(x,y))F(y) \subset F\big(\lambda(x,y)x + (1-\lambda(x,y))y\big) + K \tag{6}$$

for all $x, y \in D$. Clearly, K-convex set-valued maps are (K,λ)-convex with every function λ; K-midconvex set-valued maps are (K,λ)-convex with the constant function $\lambda = 1/2$. For real-valued functions and $K = [0,\infty)$ condition (6) reduces to

$$F\big(\lambda(x,y)x + (1-\lambda(x,y))y\big) \le \lambda(x,y)F(x) + (1-\lambda(x,y))F(y), \quad x, y \in D.$$

Such functions were introduced and discussed by Zs. Páles in [21], who obtained a Bernstein–Doetsch-type theorem for them. The next result is a set-valued generalization of this theorem.

Theorem 15. ([1, *Thm. 1*]). *Let $D \subset \mathbb{R}^n$ be an open convex set, $\lambda : D^2 \to (0,1)$ be a function continuous in each variable, Y be a locally convex space and K be a closed convex cone in Y. If a s.v. map $F : D \to c(Y)$ is (K,λ)-convex and locally K-upper bounded at a point of D, then it is K-convex.*

Finally we present a Sierpiński-type theorem for (K, λ)-convex s.v. maps.

Theorem 16. ([1, *Thm. 2*]). *Let* Y, K, *and* D *be such as in Theorem 15 and* $\lambda : D^2 \to (0,1)$ *be a continuously differentiable function. If a s.v. map* $F : D \to c(Y)$ *is* (K, λ)*-convex and Lebesgue measurable, then it is also* K*-convex.*

References

[1] M. ADAMEK, K. NIKODEM AND Z. PÁLES, On (K, λ)-convex set-valued maps, submitted.

[2] J.P. AUBIN AND A. CELLINA, *Differential Inclusions*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[3] A. AVERNA AND T. CARDINALI, Sui concetti di K–convessità (K–concavità) e di K–convessità* (K–concavità*), *Riv.Mat.Univ.Parma*, **16**(4) (1990), 311–330.

[4] J.M. BORWEIN, Multivalued convexity and optimization: A unified approach to inequality and equality constrains, *Math. Programming*, **13** (1977), 183–199.

[5] J.M. BORWEIN, Convex relations in analysis and optimization, in: *Generalized Convexity in Optimization and Economics*, Academic Press, New York, (1981), 335–377.

[6] J.M. BORWEIN, A Lagrange multiplier theorem and sandwich theorems for convex relations, *Math. Scand.*, **48** (1981), 189–204.

[7] W.W. BRECKNER Continuity of generalized convex and generalized concave set–valued functions, *Révue d'Analyse Numérique et de la Théorie de l'Approximation*, **22** (1993), 39–51.

[8] T. CARDINALI AND F. PAPALINI, Una estensione del concetto di midpoint convessitàper multifunzioni, *Riv. Mat. Univ. Parma*, **15** (1989), 119–131.

[9] A. FIACCA AND S. VERCILLO, On the K–continuity of K–hull midconvex set–valued functions, *Le Matematiche*, **48** (1993), 263–271.

[10] R. GER, Convex transformations with Banach lattice range, *Stochastica*, **11**(1) (1987), 13–23.

[11] R. GER, Mazur's criterion for continuity of convex functionals, *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, **43**(3) (1995) 263–268.

[12] M. KUCZMA, An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality, *PWN–Uniwersytet Śląski*, Warszawa–Kraków–Katowice, 1985.

[13] D. KUROIWA, T. TANAKA AND T.X.D. HA On cone convexity of set–valued maps, *Nonlinear Analysis,Methods and Appl.*, **30** (1997), 1487–1496.

[14] I. LABUDA AND R.D. MAULDIN, Problem 24 of the "Scottish Book" concerning additive functionals, *Colloquium Math.*, **48** (1984) 89–91.

[15] R.D. MAULDIN, (Ed.), *The Scottish Book, Mathematics from the Scottish Café*, Birkhauser, Basel, 1981.

[16] K. NIKODEM, Continuity of K–convex set–valued functions, *Bull. Acad. Polon. Sci. Ser. Sci. Math.*, **34** (1986), 396–400.

[17] K. NIKODEM, K–convex and K–concave set–valued functions, *Zeszyty Nauk. Politech. Łódz. Mat.*, *559*; *Rozprawy Mat.*, **114**., Łódź, 1989.

[18] K. NIKODEM, Remarks on K–midconvex set–valued functions with closed epigraph, *Le Matematiche*, **45** (1990), 277–281.

[19] K. NIKODEM, Continuity properties of midconvex set–valued maps, *Aequationes Math.*, **62** (2001), 175–183.

[20] K. NIKODEM, Mazur's criterion for continuity of convex set–valued maps, *Rocznik Nauk.-Dydakt. Akad. Pedagog. Kraków*, **204**, *Prace Mat.*,**17** (2000), 191–195.

[21] Z. PÁLES, Bernstein-Doetsch-type results for general functional inequalities, *Rocznik Nauk.-Dydakt. Akad. Pedagog. w Krakowie*, **204**, *Prace Mat.*, **17** (2000), 197–206.

[22] F. PAPALINI, La K–midpoint* canvessità (concavità) e la K–semicontinuità inferiore (superiore) di una multifunzione, *Riv.Mat.Univ.Parma*, **16**(4) (1990), 149–159.

[23] D. POPA, Semicontinuity of locally p–convex and locally p–concave set–valued maps, *Mathematica*, **41**(64) (1999).

[24] B. RICCERI, On multifunctions with convex graph, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **77** (1984), 64–70.

[25] B. RICCERI, Remarks on multifunctions with convex graph, *Arch. Math.* (Basel), **52** (1989), 519–520.

[26] A.W. ROBERTS AND D.E. VARBERG, Convex Functions, Academic Press, New York–London, 1973.

[27] S.M. ROBINSON, Regularity and stability for convex multivalued functions, *Math. Oper. Res.*, **1** (1976), 130–143.

[28] W. SONG, Duality in set-valued optimization, Dissertationes Math. (Rozprawy Mat.) 375, Warszawa 1998.

[29] A. STERNA-KARWAT, Convexity of the optimal multifunctions and its consequences in vector optimization, *Optimization*, **20** (1989), 799–808.

[30] L. THIBAULT, Continuity of measurable convex multifunctions, in: *Multifunctions and Integrands*, Springer–Verlag, *Lecture Notes in Math.*, 1091, 1983.

[31] C. URSESCU, Multifunctions with closed convex graph, *Czechoslovak Math. J.*, **25** (1975), 438–441.