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NOTE ON BERNSTEIN'S INEQUALITY FOR THE THIRD DERIVATIVE OF A POLYNOMIAL

CLÉMENT FRAPPIER

DÉPARTEMENT DE MATHÉMATIQUES ET DE GÉNIE INDUSTRIEL
ÉCOLE POLYTECHNIQUE DE MONTRÉAL,
C.P. 6079, SUCC. CENTRE-VILLE
MONTRÉAL (QUÉBEC) H3C 3A7
CANADA.
clement.frappier@polymtl.ca

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ABSTRACT. Given a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, we give the best possible constant $c_3(n)$ such that $\|p'''\| + c_3(n)|a_0| \le n(n-1)(n-2)\|p\|$, where $\| \; \|$ is the maximum norm on the unit circle $\{z : |z| = 1\}$. Most of the computations are done with a computer.

Key words and phrases: Bernstein's inequality, Unit circle.

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1. INTRODUCTION

Let $\mathcal{P}_n$ denote the class of all polynomials $p(z) = \sum_{j=0}^{n} a_j z^j$, of degree $\le n$ with complex coefficients. The famous Bernstein's inequality states that

$$\|p'\| \le n\|p\|, \tag{1.1}$$

where $\|p\| := \max_{|z|=1} |p(z)|$. The inequality (1.1) has been refined and generalized in numerous ways; see [3] for many interesting results. It is obvious from (1.1) that

$$\|p^{(k)}\| \le \frac{n!}{(n-k)!}\|p\| \tag{1.2}$$

for $1 \le k \le n$. Let $c_k(n)$ be the *best possible* constant such that

$$\|p^{(k)}\| + c_k(n)|a_0| \le \frac{n!}{(n-k)!}\|p\|. \tag{1.3}$$

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By "best possible" we mean that, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(z) = \sum_{j=0}^n a_j(\varepsilon) z^j$ such that

$$\|p_\varepsilon^{(k)}\| + (c_k(n) + \varepsilon)\,|a_0(\varepsilon)| > \frac{n!}{(n-k)!}\|p_\varepsilon\|.$$

It is known (see [4, p. 125] or [2, p. 70]) that $c_1(1) = 1$ and $c_1(n) = \frac{2n}{n+2}$, $n \geq 2$. We [1, p. 30] also have $c_2(n) = \frac{2(n-1)(2n-1)}{(n+1)}$, $n \geq 1$. The aim of this note is to prove the following result.

Theorem 1.1. *Let $p \in \mathcal{P}_n$, $p(z) = \sum_{j=0}^n a_j z^j$. If we denote by $c_3(n)$ the best possible constant such that*

$$\|p'''\| + c_3(n)|a_0| \leq n(n-1)(n-2)\|p\| \tag{1.4}$$

then $c_3(1) = c_3(2) = 0$ and, for $n \geq 3$,

$$c_3(n) = \frac{A(n)}{B(n)}, \tag{1.5}$$

where

$$\begin{aligned} A(n) := 6(n-2)(n-1)^3\Bigg((n-1)^3(8n^2-15n+6) \\ + (n-1)^2(2n^3+7n^2-21n+6)T_n\left(\frac{n(n-2)}{(n-1)^2}\right) \\ - n(11n^3-47n^2+56n-14)U_n\left(\frac{n(n-2)}{(n-1)^2}\right)\Bigg)\end{aligned}$$

and

$$\begin{aligned} B(n) := n\Bigg(4(n-1)^5(2n-1) \\ + 2(n-1)^2(n^4+3n^3-13n^2+10n-2)T_n\left(\frac{n(n-2)}{(n-1)^2}\right) \\ - n(11n^4-54n^3+86n^2-50n+9)U_n\left(\frac{n(n-2)}{(n-1)^2}\right)\Bigg).\end{aligned}$$

Here, $T_n(x) := \cos(n \arccos(x))$ and $U_n(x) := \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))}$ are respectively the Chebyshev polynomials of the first and second kind.

2. Proof of the Theorem

The method of proof used to obtain inequalities of the type (1.3) is well described in the aforementioned references. We give some details for the sake of completeness. Consider two analytic functions

$$f(z) = \sum_{j=0}^\infty a_j z^j, \quad g(z) = \sum_{j=0}^\infty b_j z^j$$

for $|z| \leq K$. The function

$$(f \star g)(z) := \sum_{j=0}^\infty a_j b_j z^j \quad (|z| \leq K)$$

is said to be their Hadamard product.

Let $\mathcal{B}_n$ be the class of polynomials Q in $\mathcal{P}_n$ such that

$$\|Q \star p\| \leq \|p\| \quad \text{for every } p \in \mathcal{P}_n.$$

To $p \in \mathcal{P}_n$ we associate the polynomial $\widetilde{p}(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Observe that

$$Q \in \mathcal{B}_n \iff \widetilde{Q} \in \mathcal{B}_n.$$

Let us denote by $\mathcal{B}_n^0$ the subclass of $\mathcal{B}_n$ consisting of polynomials R in $\mathcal{B}_n$ for which $R(0) = 1$. The following lemma contains a useful characterization of $\mathcal{B}_n^0$.

Lemma 2.1. [2] *The polynomial* $R(z) = \sum_{j=0}^{n} b_j z^j$, *where* $b_0 = 1$, *belongs to* $\mathcal{B}_n^0$ *if and only if the matrix*

$$M(b_0, b_1, \ldots, b_n) := \begin{pmatrix} b_0 & b_1 & \ldots & b_{n-1} & b_n \\ \bar{b}_1 & b_0 & \ldots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{n-1} & \bar{b}_{n-2} & \ldots & b_0 & b_1 \\ \bar{b}_n & \bar{b}_{n-1} & \ldots & \bar{b}_1 & b_0 \end{pmatrix}$$

is positive semi-definite.

The following well-known result enables us to study the definiteness of the matrix $M(1, b_1, \ldots, b_n)$ associated with the polynomial $R(z) = \widetilde{Q}(z) = 1 + \sum_{j=1}^{n} b_j z^j$.

Lemma 2.2. *The hermitian matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{12} & \ldots & a_{nn} \end{pmatrix}, \quad a_{ij} = \bar{a}_{ji},$$

is positive definite if and only if its leading principal minors are all positive.

Here we simply use the calculations done in [1], where Lemmas 2.1 and 2.2 are applied to a polynomial of the form

$$R(z) = 1 + \sum_{j=1}^{n-1} \frac{j(j-1)(j-2)}{n(n-1)(n-2)} z^j + \frac{\alpha}{n(n-1)(n-2)} z^n.$$

In that paper, the evaluation of the best possible constant $c_3(n)$ is reduced to the evaluation of the least positive root of the quadratic polynomial in c

$$(2.1) \qquad D(n,c) := \begin{vmatrix} -c & -6(n-1)^2 & 12n(n-2) & -6(n-1)^2 & 0 \\ 6+c & x_{1,n-4} & y^*_{1,n-4} & 6n(n-2) & -6-\frac{c}{(n-1)^2} \\ -c & x_{2,n-4} & y_{2,n-4} & -6(n-1)(n-2) & 6(n-2)-\frac{c}{(n-1)^2} \\ c & x_{3,n-5} & y_{3,n-5} & 3(n-1)(n-2) & -3(n-1)(n-2) \\ -c & x_{4,n-6} & y_{4,n-6} & (n-1)(n-2)(n-3) & n(n-1)(n-2) \end{vmatrix},$$

where

$$\begin{gathered} x_{j,1} = H_{j,1} + \tfrac{12n(n-2)}{(n-1)^2}, \\ x_{j,2} = y_{j,1} + \frac{2n(n-2)}{(n-1)^2} x_{j,1}, \quad y_{j,1} = H_{j,1} - 6, \\ x_{j,k} - \frac{2n(n-2)}{(n-1)^2} x_{j,k-1} + x_{j,k-2} = H_{j,k}, \\ y_{1,k} + x_{1,k-1} = 6 \quad \text{for} \quad 2 \le k \le n-5, \\ y_{j,k} + x_{j,k-1} = H_{j,k}, \\ y^*_{1,n-4} + x_{1,n-5} = -6n(n-2) \end{gathered}$$

for $j = 1, 2, 3, 4$, and

$$H_{j,k} = \begin{cases} 6 & \text{if } j = 1, \quad 1 \le k \le n-4, \\ 6(k+1) & \text{if } j = 2, \quad 1 \le k \le n-4, \\ 3(k+1)(k+2) & \text{if } j = 3, \quad 1 \le k \le n-5, \\ (k+1)(k+2)(k+3) & \text{if } j = 4, \quad 1 \le k \le n-6. \end{cases}$$

It was impossible at the time of [1] to obtain a simple expression for $D(n, c)$. With the development of mathematical software, it has become possible to handle nearly all the difficulties. The following computations can be done with Mathematica 4.1.

The determinant $D(n, c)$ can be expressed in the form

$$D(n, c) = q_0(n) - q_1(n)c - q_2(n)c^2, \tag{2.2}$$

where

$$\begin{aligned} q_0(n) := \frac{81(n-2)(n-1)^8}{(2n^2-4n+1)} \Bigg(& 8(n-1)(2n-3)(2n^2-4n+1) \\ & + (n-1)^{-2n}\big(n(n-2) - i\sqrt{2n^2-4n+1}\big)^n \\ & \quad \times \big(2(2n^2-4n+1)(n^2+2n-6) \\ & - i(n-2)(11n^2-20n+3)\sqrt{2n^2-4n+1}\big) \\ & + (n-1)^{-2n}\big(n(n-2) \\ & + i\sqrt{2n^2-4n+1}\big)^n\big(2(2n^2-4n+1)(n^2+2n-6) \\ & + i(n-2)(11n^2-20n+3)\sqrt{2n^2-4n+1}\big)\Bigg), \end{aligned} \tag{2.3}$$

$$\begin{aligned} q_1(n) := \frac{54(n-1)^5}{(2n^2-4n+1)} \Bigg(& (n-1)(7n^2-14n+6)(2n^2-4n+1) \\ & + (n-1)^{-2n}\big(n(n-2) - i\sqrt{2n^2-4n+1}\big)^n \\ & \quad \times (5n^2-10n+3)\big((2n^2-4n+1) \\ & - in(n-2)\sqrt{2n^2-4n+1}\big) + (n-1)^{-2n}\big(n(n-2) \\ & + i\sqrt{2n^2-4n+1}\big)^n(5n^2-10n+3)\big((2n^2-4n+1) \\ & + in(n-2)\sqrt{2n^2-4n+1}\big)\Bigg), \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
(2.5)\qquad q_2(n) := \frac{9n(n-1)^2}{4(2n^2-4n+1)}\Bigg(& 8(n-1)^3(2n-1) + (n-1)^{-2n}\big(n(n-2) \\
& - i\sqrt{2n^2-4n+1}\big)^n\big(2(2n^2-4n+1) \\
& \quad \times (n^4+3n^3-13n^2+10n-2) \\
& - in(11n^4-54n^3+86n^2-50n+9)\sqrt{2n^2-4n+1}\big) \\
& + (n-1)^{-2n}\big(n(n-2) \\
& + i\sqrt{2n^2-4n+1}\big)^n\big(2(2n^2-4n+1) \\
& \quad \times (n^4+3n^3-13n^2+10n-2) \\
& + in(11n^4-54n^3+86n^2-50n+9)\sqrt{2n^2-4n+1}\big)\Bigg).
\end{aligned}
$$

The only real problem we encountered was that the software was unable to recognize that the discriminant $q_1^2 + 4q_0q_2$ is a perfect square. It is necessary to observe that

$$
\begin{aligned}
(2.6)\quad q_1^2 + 4q_0q_2 = \Bigg(& \frac{27(n-1)^{-2n+9}}{\sqrt{2n^2-4n+1}}\big(4(2n-1)(2n-3)(n-1)^{2(n-1)}\sqrt{2n^2-4n+1} \\
& + \big(2n(n-2)\sqrt{2n^2-4n+1} - i(n-1)(11n^2-22n+6)\big)\big(n(n-2) \\
& - i\sqrt{2n^2-4n+1}\big)^{n-1} + \big(2n(n-2)\sqrt{2n^2-4n+1} \\
& + i(n-1)(11n^2-22n+6)\big)\big(n(n-2) + i\sqrt{2n^2-4n+1}\big)^{n-1}\big)\Bigg)^2.
\end{aligned}
$$

The positive root of $D(n,c)$ is readily found with the help of (2.6). That root can be written in the form (1.5) where $e^{it} := \frac{n(n-2)-i\sqrt{2n^2-4n+1}}{(n-1)^2}$. Throughout the calculations, the identity

$$\left(n(n-2) - i\sqrt{2n^2-4n+1}\right)^n \left(n(n-2) + i\sqrt{2n^2-4n+1}\right)^n = (n-1)^{4n}$$

is useful for simplifying the expressions.

3. TWO OPEN QUESTIONS

We immediately obtain the values $c_3(3) = 6$, $c_3(4) = \frac{156}{7}$, $c_3(5) = \frac{5736}{115}$, $c_3(6) = \frac{92955}{1043}$, $c_3(7) = \frac{342430}{2443}$, etc. It is highly probable that all the constants $c_k(n)$, appearing in (1.3), are rational numbers. Other values are $c_4(4) = 24$, $c_4(5) = \frac{2184}{19}$, $c_4(6) = \frac{11808}{37}$, $c_4(7) = \frac{11625}{17}$, etc. In fact, all the constants $c_{k,\nu}(n)$, related to the same kind of inequality with $|a_\nu|$ rather than $|a_0|$, seem to be rational numbers.

Question 1. Are the constants $c_{k,\nu}(n)$ rational numbers? □

As far as the constants $c_k(n)$, $k \geq 4$, are concerned, it is probable that a few simple expressions can be found for them. The most interesting problem here is perhaps to find the asymptotic value of $c_k(n)$ as $n \to \infty$. We have

$$(3.1)\qquad \lim_{n\to\infty} \frac{c_k(n)}{n^{k-1}} = 2k$$

for $k = 0, 1, 2$ and $k = 3$. The latter case follows from (1.5).

Question 2. Does (3.1) hold for $k \geq 4$? □

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