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ON THE ITERATED GREEN FUNCTIONS ON A BOUNDED DOMAIN AND THEIR RELATED KATO CLASS OF POTENTIALS

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ABSTRACT. We use the results of Zhang [15, 16] and Davies [7] on the behavior of the heat kernel $p(t,x,y)$ on a bounded $C^{1,1}$ domain D to find again the result of Grunau-Sweers [9] concerning the estimates of the iterated Greens functions $G_{m,n}(D)$. Next, we use these estimates to characterize, by means of $p(t,x,y)$, the Kato class $K_{m,n}(D)$ and we give new examples of functions belonging to this class.

Key words and phrases: Green function, Gauss semigroup, Kato class.

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1. INTRODUCTION

Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 3$ and $p\,(t,x,y)$ be the density of the Gauss semigroup on D. Combining the results of Zhang [15], [16] and those of Davies or Davies-Simon [7], [8] a qualitatively sharp understanding of the boundary behaviour of $p\,(t,x,y)$ is given as follows: There exist positive constants c_1 , c_2 and λ_0 depending only on D such that for all $t > 0$ and x , $y \in D$,

$$\left(\frac{\delta(x)}{\sqrt{t}\wedge 1}\wedge 1\right)\left(\frac{\delta(y)}{\sqrt{t}\wedge 1}\wedge 1\right)\frac{c_1 e^{-\lambda_0 t - c_2\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}} \leq p\,(t,x,y) \leq \left(\frac{\delta(x)}{\sqrt{t}\wedge 1}\wedge 1\right)\left(\frac{\delta(y)}{\sqrt{t}\wedge 1}\wedge 1\right)\frac{e^{-\lambda_0 t - \frac{|x-y|^2}{c_2 t}}}{c_1 t^{\frac{n}{2}}}, \tag{1.1}$$

where $\delta(x)$ denotes the Euclidean distance from x to the boundary of D.

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Let $G(x,y)$ be the Green's function of the laplacien Δ in D with a Dirichlet condition on ∂D. Then G is given by

$$G(x,y)=\int_0^\infty p\,(t,x,y)\,dt,\quad \text{for } x\,,y\in D. \tag{1.2}$$

For a positive integer m, we denote by $G_{m,n}$ the Green's function of the operator $u\mapsto(-\Delta)^m u$ on D with Navier boundary conditions $\Delta^j u=0$ on ∂D for $0\le j\le m-1$. Then $G_{1,n}=G$ and $G_{m,n}$ satisfies for $m\ge 2$

$$G_{m,n}(x,y)=\int_D\int_D G(x,z)G_{m-1,n}(z,y)\,dz.$$

Using the Fubini theorem and the Chapman-Kolmogorov identity, we show by induction that for each $m\ge 1$ and $x\,,y\in D$ we have

$$G_{m,n}(x,y)=\frac{1}{(m-1)!}\int_0^\infty t^{m-1}p\,(t,x,y)\,dt. \tag{1.3}$$

In this paper we will use (1.1) and (1.3) to find again the result of Grunau and Sweers in [9] concerning the sharp estimates of $G_{m,n}$. More precisely we will give another proof for the case $n\ge 3$ of the following theorem.

Theorem 1.1 (see [9]). *On D^2 we have*

$$G_{m,n}(x,y)\sim H_{m,n}(x,y)=\begin{cases}\frac{1}{|x-y|^{n-2m}}\min\left(1\,,\frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \textit{if } n>2m,\\ \operatorname{Log}\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \textit{if } n=2m,\\ \sqrt{\delta(x)\delta(y)}\,\min\left(1\,,\frac{\sqrt{\delta(x)\delta(y)}}{|x-y|}\right) & \textit{if } n=2m-1,\\ \delta(x)\delta(y)\operatorname{Log}\left(2+\frac{1}{|x-y|^2+\delta(x)\delta(y)}\right) & \textit{if } n=2m-2,\\ \delta(x)\delta(y) & \textit{if } n<2m-2,\end{cases}$$

where the symbol $\sim$ is defined in the notations below.

As a second step we will also use (1.1) and (1.3) to give new contributions in the case $n>2m$ to the study of the Kato class $K_{m,n}(D)$ defined in [11] for $m=1$ and in [2] for $m\ge 2$ as follows.

Definition 1.1. A Borel measurable function q in D belongs to the Kato class $K_{m,n}(D)$ if q satisfies the following condition

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{D\cap(|x-y|\le\alpha)}\frac{\delta(y)}{\delta(x)}G_{m,n}(x,y)|q(y)|\,dy=0. \tag{1.4}$$

We note that in the case $m=1$, the class $K_{1,n}(D)$ properly contains the classical Kato class $K_n(D)$ introduced in [1] as the natural class of singular functions which replaces the L^p-Lebesgue spaces in order that the weak solutions of the Shrödinger equation are continuous and satisfy a Harnack principle. More precisely, it is shown in [11] that the function $\rho_\alpha(y)=\frac{1}{\delta^\alpha(y)}$ belongs to $K_{1,n}(D)$ if and only if $\alpha<2$ but for $1\le\alpha<2,\ \rho_\alpha\notin K_n(D)$.

Our second contribution here is to exploit estimates of Theorem 1.1 on the one hand, to give new examples of functions belonging to the class $K_{m,n}(D)$ and to characterize this class by means of the density of the Gauss semigroup in D on the other hand. In particular we will prove the following results for the unit ball.

Proposition 1.2. *For $\lambda\,,\,\mu\in\mathbb{R}$ and $y\in B(0,1)$ we put*

$$\rho_{\lambda,\,\mu}(y)=\frac{1}{(1-|y|)^{\lambda}\left[\mathrm{Log}(\frac{2}{1-|y|})\right]^{\mu}}.$$

For $m\geq 2$ we have

$$\rho_{\lambda,\mu}\in K_{m,n}(B(0,1))\ \textit{ if and only if }\quad \lambda<3 \textit{ or } (\lambda=3 \textit{ and } \mu>1).$$

Theorem 1.3. *Let $n>2m$ and q be a Borel measurable function in D. Then the following assertions are equivalent:*

1) $q\in K_{m,n}(B(0,1))$
2) $\lim_{t\to 0}\left(\sup_{x\in B}\int_0^t\int_B\frac{\delta(y)}{\delta(x)}\,s^{m-1}p(s,x,y)|q(y)|\,dyds\right)=0$

We also note that in the case $m=1$, similar characterizations have been obtained by Aizenman and Simon in [1] for the Kato class $K_n(\mathbb{R}^n)$ and by Bachar and Mâagli in [4] for the half space $\mathbb{R}^n_+$, where they introduce a new Kato class that properly contains the classical one. This was extended for $m\geq 2$ by Mâagli and Zribi [12] to the class $K_{m,n}(\mathbb{R}^n)$ and by Bachar [3] to the class $K_{m,n}(\mathbb{R}^n_+)$. The density of the Gauss semigroup in the case of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ are explicitly known, but this is not the case for a bounded $C^{1,1}$ domain even if D is an open ball .

In order to simplify our statements, we define some convenient notations.

Notations.

i) For $x,y\in D$, we denote by $\delta(x)$ the Euclidean distance from x to the boundary of D, $[x,y]^2=|x-y|^2+\delta(x)\delta(y)$ and d is the diameter of D.

ii) For $a,b\in\mathbb{R}$, we denote by $a\wedge b=\min(a,b)$ and $a\vee b=\max(a,b)$.

iii) Let f and g be two nonnegative functions on a set S.
We say that $f\preceq g$, if there exists $c>0$ such that

$$f(x)\leq c\,g(x)\quad\text{for all } x\in S.$$

We say that $f\sim g$, if there exists $C>0$ such that

$$\frac{1}{C}g(x)\leq f(x)\leq Cg(x)\quad\text{for all } x\in S.$$

The following properties will be used several times

iv) For $a,b\geq 0$, we have

$$\frac{a\,b}{a+b}\leq\min(a,b)\leq 2\,\frac{a\,b}{a+b}\tag{1.5}$$

$$(a+b)^p\sim a^p+b^p\quad\text{for } p\in\mathbb{R}^+.\tag{1.6}$$

$$\min(1,a)\min(1,b)\leq\min(1,a\,b)\leq\min(1,a)\max(1,b)\tag{1.7}$$

$$\frac{a}{1+a}\leq\mathrm{Log}(1+a)\tag{1.8}$$

v) Let $\eta,\nu>0$ and $0<\gamma\leq 1$. Then we have

$$\mathrm{Log}(1+t)\preceq t^{\gamma}\,,\quad\text{for } t\geq 0.\tag{1.9}$$

$$\mathrm{Log}(1+\eta\,t)\sim\mathrm{Log}(1+\nu\,t)\,,\quad\text{for } t\geq 0.\tag{1.10}$$

Finally we note that since for each $a \geq b \geq 0$ and $c > 0$ we have

$$\begin{aligned}\frac{(a+1)(b+1)}{1+ab}\, e^{-c\,(b-a)^2} &= \left(1+\frac{a+b}{1+ab}\right) e^{-c\,(b-a)^2} \\ &= \left(1+\frac{2a+\xi}{1+a(a+\xi)}\right) e^{-c\xi^2} \\ &\leq (2+\xi)e^{-c\xi^2} \leq C.\end{aligned}$$

Then, using (1.5) we deduce that for each $x\,, y \in D$ and $0 < t \leq 1$ we have

$$\begin{aligned}\min\left(\frac{\delta(x)\delta(y)}{t}\,,1\right) &\leq C \min\left(\frac{\delta(x)}{\sqrt{t}},1\right) \min\left(\frac{\delta(y)}{\sqrt{t}},1\right) e^{c\,\frac{|\delta(x)-\delta(y)|^2}{t}} \\ &\leq C \min\left(\frac{\delta(x)}{\sqrt{t}},1\right) \min\left(\frac{\delta(y)}{\sqrt{t}},1\right) e^{c\,\frac{|x-y|^2}{t}}.\end{aligned}$$

So, using this fact, (1.7) and the fact that D is bounded we deduce that estimates (1.1) can be written as follows:

There exist positive constants c, C and λ such that

$$\frac{1}{C}\, h_{\frac{1}{c},\lambda}(t,x,y) \leq p\,(t,x,y) \leq C\, h_{c,\lambda}(t,x,y), \tag{1.11}$$

where

$$h_{c,\lambda}(t,x,y) := \begin{cases} \min\left(\frac{\delta(x)\delta(y)}{t},1\right) \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}}\,, & \text{if } 0 < t \leq 1 \\ \delta(x)\delta(y)e^{-\lambda t}\,, & \text{if } t > 1. \end{cases} \tag{1.12}$$

Throughout the paper, the letter C will denote a generic positive constant which may vary from line to line.

2. PROOF OF THEOREM 1.1

First we need the following lemma.

Lemma 2.1. *For each* $x, y \in D$ *we have*

$a)$ *For* $n \geq 2m$

$$\delta(x)\delta(y) \leq \min\left(1\,, \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \frac{d^{n-2m+2}}{|x-y|^{n-2m}}.$$

$b)$

$$\delta(x)\delta(y) \leq d^2\, \min\left(1\,, \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \leq 2d^2\, \mathrm{Log}\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right).$$

Now we will give the proof of Theorem 1.1. More precisely, using (1.3) and (1.11) we will prove that for each $c > 0$, we have

$$\int_0^\infty t^{m-1} h_{c,\lambda}(t,x,y)dt \sim H_{m,n}(x,y)\,.$$

Without loss of generality we will assume that $\lambda = 1\,,\ c = 1$ and denote by $h_{1,1}(t,x,y) = h(t,x,y)$. Hence, using a change of variable, we obtain

$$\int_0^\infty t^{m-1} h(t,x,y)dt = C\,\delta(x)\delta(y) + \int_0^1 t^{m-1} \min\left(\frac{\delta(x)\delta(y)}{t}, 1\right) \frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}}\, dt$$
$$= C\,\delta(x)\delta(y) + |x-y|^{2m-n} \int_{|x-y|^2}^\infty r^{\frac{n}{2}-m-1} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\, r,\ 1\right) e^{-r}\, dr.$$

Since we will sometimes omit e^{-r} and we need to integrate the functions $r \to r^{\frac{n}{2}-m-1}$ and $r \to r^{\frac{n}{2}-m}$ near zero or near ∞, we will discuss the following cases

Case 1. $n > 2m$. Using (1.7) we obtain

$$\min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\,,\ 1\right) \min(r,1) \leq \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\, r,\ 1\right)$$
$$\leq \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\,,\ 1\right) \max(r\,,\ 1)).$$

Hence the lower bound follows from the fact that

$$\int_{|x-y|^2}^\infty \min(1,r) r^{\frac{n}{2}-m-1} e^{-r}\, dr \geq \int_{d^2}^\infty \min(1,r) r^{\frac{n}{2}-m-1} e^{-r}\, dr = C$$

and the upper bound follows from Lemma 2.1.

Case 2. $n = 2m$. In this case

$$\int_0^\infty t^{m-1} h(t,x,y)dt = C\,\delta(x)\delta(y) + \int_{|x-y|^2}^\infty \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\,, \frac{1}{r}\right) e^{-r}\, dr.$$

So using (1.5) and the fact that

$$\frac{|x-y|^2 + (2d^2+1)\delta(x)\delta(y)}{1+\delta(x)\delta(y)} \geq |x-y|^2 + \delta(x)\delta(y)\,,$$

we obtain

$$\int_0^\infty t^{m-1} h(t,x,y)dt \geq \int_{|x-y|^2}^{2d^2+1} \frac{\delta(x)\delta(y)}{|x-y|^2 + r\delta(x)\delta(y)}\, dr$$
$$= C\,\mathrm{Log}\left(\frac{|x-y|^2 + (2d^2+1)\delta(x)\delta(y)}{|x-y|^2(1+\delta(x)\delta(y))}\right)$$
$$\geq C\,\mathrm{Log}\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right).$$

To prove the upper inequality we use (1.5), (1.11) and (1.10) to obtain

$$\begin{aligned}
&\int_{|x-y|^2}^{\infty} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\,,\,\frac{1}{r}\right) e^{-r} dr \\
&\leq C \int_{|x-y|^2}^{\infty} \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r+|x-y|^2}\, e^{-r} dr \\
&\leq C \int_{|x-y|^2}^{d^2+1} \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r+|x-y|^2}\, dr + C\frac{\delta(x)\delta(y)}{[x,y]^2} \int_{1+d^2}^{\infty} e^{-r} dr \\
&= C \operatorname{Log}\left(\frac{|x-y|^2+(d^2+1)\delta(x)\delta(y)}{|x-y|^2(1+\delta(x)\delta(y))}\right) + C\frac{\delta(x)\delta(y)}{[x,y]^2} \\
&\leq C \operatorname{Log}\left(1+\frac{(d^2+1)\delta(x)\delta(y)}{|x-y|^2(1+\delta(x)\delta(y))}\right) + C\frac{\delta(x)\delta(y)}{|x,y|^2+\delta(x)\delta(y)} \\
&\leq C \operatorname{Log}\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right).
\end{aligned}$$

Hence the result follows from Lemma 2.1.

Case 3. $n = 2m-1$. In this case

$$\begin{aligned}
&\int_0^{\infty} t^{m-1} h(t,x,y)\, dt \\
&= C\delta(x)\delta(y) + |x-y| \int_{|x-y|^2}^{\infty} r^{-\frac{1}{2}} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}, \frac{1}{r}\right) e^{-r}\, dr \\
&\leq C\,\frac{\delta(x)\delta(y)}{|x-y|}\left(d + \int_0^{\infty} r^{-\frac{1}{2}}\, e^{-r} dr\right) = C\,\frac{\delta(x)\delta(y)}{|x-y|}.
\end{aligned}$$

On the other hand, an integration by parts shows that

$$\begin{aligned}
&|x-y| \int_{|x-y|^2}^{\infty} r^{-\frac{1}{2}} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\,,\,\frac{1}{r}\right) e^{-r}\, dr \\
&\leq C\frac{\delta(x)\delta(y)}{|x-y|} \int_0^{d^2\frac{|x-y|^2}{\delta(x)\delta(y)}} r^{-\frac{1}{2}}\, dr + |x-y| \int_{d^2\frac{|x-y|^2}{\delta(x)\delta(y)}}^{\infty} r^{-\frac{3}{2}} e^{-r}\, dr \\
&\leq C\sqrt{\delta(x)\delta(y)} + |x-y| \left[-2r^{-\frac{1}{2}} e^{-r}\right]_{d^2\frac{|x-y|^2}{\delta(x)\delta(y)}}^{\infty} \\
&\leq C\sqrt{\delta(x)\delta(y)}.
\end{aligned}$$

Hence

$$\int_0^{\infty} t^{m-1} h(t,x,y) dt \leq C \min\left(\sqrt{\delta(x)\delta(y)}\,,\,\frac{\delta(x)\delta(y)}{|x-y|}\right).$$

For the lower inequality we discuss two subcases

- If $\delta(x)\delta(y) \leq |x-y|^2$. Then from (1.7) we have

$$\begin{aligned}
\int_0^{\infty} t^{m-1} h(t,x,y) dt &\geq |x-y| \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \int_{1+d^2}^{\infty} r^{-\frac{3}{2}} e^{-r} dr \\
&= C\frac{\delta(x)\delta(y)}{|x-y|}.
\end{aligned}$$

- If $|x-y|^2 \leq \delta(x)\delta(y)$. Then

$$\begin{aligned}\int_0^\infty t^{m-1}h(t,x,y)dt &\geq |x-y| \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge \frac{1}{r}\right) r^{-\frac{1}{2}} e^{-r} dr \\ &\geq C\,\frac{\delta(x)\delta(y)}{|x-y|} \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} r^{-\frac{1}{2}} e^{-r} dr \\ &\geq C\,\frac{\delta(x)\delta(y)}{|x-y|} \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} r^{-\frac{1}{2}} dr \\ &\geq C\,\delta(x)\delta(y)\left[\frac{2d}{\sqrt{\delta(x)\delta(y)}} - 1\right] \\ &\geq C\,\sqrt{\delta(x)\delta(y)}\left[2d - \sqrt{\delta(x)\delta(y)}\right] \\ &\geq C\,\sqrt{\delta(x)\delta(y)}.\end{aligned}$$

Case 4. $n = 2m-2$. In this case, we use (1.5) to deduce that

$$\begin{aligned}&\int_0^\infty t^{m-1}h(t,x,y)dt \\ &= C\delta(x)\delta(y) + |x-y|^2 \int_{|x-y|^2}^{\infty} \left(\frac{\delta(x)\delta(y)}{r|x-y|^2} \wedge \frac{1}{r^2}\right) e^{-r} dr. \\ &\sim \delta(x)\delta(y) + \delta(x)\delta(y) \int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2}\right) e^{-r} dr.\end{aligned}$$

To prove the upper estimates we remark first that

$$\delta(x)\delta(y) \leq C\,\delta(x)\delta(y)\,\mathrm{Log}\left(2 + \frac{1}{[x,y]^2}\right)$$

and we discuss the following subcases

- If $\frac{1}{2} \leq \delta(x)\delta(y)\left(1 + \frac{1}{[x,y]^2}\right)$. Then $1 + \frac{1}{\delta(x)\delta(y)} \leq 4 + \frac{2}{[x,y]^2}$. So

$$\begin{aligned}\int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2}\right) e^{-r} dr &\leq \int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2}\right) dr \\ &= \mathrm{Log}\left(1 + \frac{1}{\delta(x)\delta(y)}\right) \\ &\leq \mathrm{Log}\,2 + \mathrm{Log}\left(2 + \frac{1}{[x,y]^2}\right) \\ &\leq C\,\mathrm{Log}\left(2 + \frac{1}{[x,y]^2}\right).\end{aligned}$$

- If $\delta(x)\delta(y)\left(1+\frac{1}{[x,y]^2}\right) \leq \frac{1}{2}$. Then $\delta(x)\delta(y)\left([x,y]^2+1\right) \leq \frac{1}{2}[x,y]^2$, which implies that $\delta(x)\delta(y) \leq |x-y|^2$ and consequently $[x,y]^2 \leq 2|x-y|^2$. Hence

$$\begin{aligned}
&\int_{|x-y|^2}^{\infty}\left(\frac{1}{r}-\frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r+|x-y|^2}\right)e^{-r}dr \\
&\leq C\,\mathrm{Log}\left(1+\delta(x)\delta(y)\right)e^{-|x-y|^2}+C\int_{|x-y|^2}^{\infty}\mathrm{Log}\left(\frac{r}{\delta(x)\delta(y)r+|x-y|^2}\right)e^{-r}dr \\
&\leq C\,\mathrm{Log}(1+d^2)\,e^{-|x-y|^2}+C\int_{|x-y|^2}^{\infty}\mathrm{Log}\left(\frac{r}{\delta(x)\delta(y)r+|x-y|^2}\right)e^{-r}dr \\
&\leq C\int_{|x-y|^2}^{\infty}\mathrm{Log}\left(\frac{(1+d^2)r}{\delta(x)\delta(y)r+|x-y|^2}\right)e^{-r}dr \\
&\leq C\int_{|x-y|^2}^{\infty}\mathrm{Log}\left(\frac{(1+d^2)r}{|x-y|^2(1+\delta(x)\delta(y))}\right)e^{-r}dr \\
&\leq C\,\mathrm{Log}\left(\frac{1+d^2}{|x-y|^2}\right)+C\int_{|x-y|^2}^{\infty}\mathrm{Log}\left(\frac{1}{1+\delta(x)\delta(y)}\,r\right)e^{-r}dr \\
&\leq C\,\mathrm{Log}\left(\frac{1+d^2}{|x-y|^2}\right)+C\int_{|x-y|^2}^{\infty}\mathrm{Log}(r)e^{-r}dr \\
&\leq C\,\mathrm{Log}\left(\frac{1+d^2}{|x-y|^2}\right)+C\int_{1}^{\infty}\mathrm{Log}(r)e^{-r}dr \\
&\leq C+C\,\mathrm{Log}\left(\frac{1+d^2}{|x-y|^2}\right)\leq C\,\mathrm{Log}\left(2+\frac{1}{[x,y]^2}\right).
\end{aligned}$$

Hence

$$\int_0^{\infty} t^{m-1}h(t,x,y)dt \leq C\,\delta(x)\delta(y)\,\mathrm{Log}\left(2+\frac{1}{[x,y]^2}\right).$$

Next we prove the lower estimates.

$$\begin{aligned}
\int_0^{\infty} t^{m-1}h(t,x,y)dt &\sim \delta(x)\delta(y)+\delta(x)\delta(y)\int_{|x-y|^2}^{\infty}\left(\frac{1}{r}-\frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r+|x-y|^2}\right)e^{-r}dr \\
&\geq C\delta(x)\delta(y)+C\delta(x)\delta(y)\int_{|x-y|^2}^{2d^2}\left(\frac{1}{r}-\frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r+|x-y|^2}\right)dr \\
&= C\delta(x)\delta(y)+C\delta(x)\delta(y)\,\mathrm{Log}\left(\frac{2d^2(1+\delta(x)\delta(y))}{|x-y|^2+2d^2\delta(x)\delta(y)}\right).
\end{aligned}$$

Let $\alpha > 1$ such that $\alpha\frac{2d^2}{2d^2+1} > 2[x,y]^2+1$; $\forall x,y \in D$. Then we have

$$\frac{2\alpha\, d^2(1+\delta(x)\delta(y))}{|x-y|^2+2d^2\delta(x)\delta(y)} \geq \frac{2\alpha\, d^2}{(1+2d^2)[x,y]^2} \geq 2+\frac{1}{[x,y]^2}.$$

Hence

$$\begin{aligned}
&\int_0^{\infty} t^{m-1}h(t,x,y)dt \\
&\geq C\delta(x)\delta(y)\left[\operatorname{Log}\alpha + \operatorname{Log}\left(\frac{2d^2(1+\delta(x)\delta(y))}{|x-y|^2+2d^2\delta(x)\delta(y)}\right)\right] \\
&\geq C\delta(x)\delta(y)\operatorname{Log}\left(\frac{2\alpha\, d^2(1+\delta(x)\delta(y))}{|x-y|^2+2d^2\delta(x)\delta(y)}\right) \\
&\geq C\delta(x)\delta(y)\operatorname{Log}\left(2+\frac{1}{[x,y]^2}\right).
\end{aligned}$$

Case 5. $n < 2m-2$. In this case we need only to prove the upper inequality.

$$\begin{aligned}
&|x-y|^{2m-n}\int_{|x-y|^2}^{\infty} r^{\frac{n}{2}-m}\min\left(\frac{\delta(x)\delta(y)}{|x-y|^2},\frac{1}{r}\right)e^{-r}\,dr \\
&\leq \delta(x)\delta(y)|x-y|^{2m-n-2}\int_{|x-y|^2}^{d^2\frac{|x-y|^2}{\delta(x)\delta(y)}} r^{\frac{n}{2}-m}dr + |x-y|^{2m-n}\int_{d^2\frac{|x-y|^2}{\delta(x)\delta(y)}}^{\infty} r^{\frac{n}{2}-m-1}dr \\
&\leq \frac{2}{2m-n-2}\,\delta(x)\delta(y)\left[1-\left(\frac{\delta(x)\delta(y)}{d^2}\right)^{m-1-\frac{n}{2}}\right] + \frac{2}{2m-n}\left(\frac{\delta(x)\delta(y)}{d^2}\right)^{m-\frac{n}{2}} \\
&\leq \frac{2}{2m-n-2}\,\delta(x)\delta(y) + \frac{2}{d^2(2m-n)}\left(\frac{\delta(x)\delta(y)}{d^2}\right)^{m-1-\frac{n}{2}}\delta(x)\delta(y) \\
&\leq C\,\delta(x)\delta(y).
\end{aligned}$$

This completes the proof of the theorem. □

Now using estimates of Theorem 1.1 and similar arguments as in the proof of Corollary 2.5 in [5], we obtain the following.

Corollary 2.2. *Let $r_0 > 0$. For each $x,y \in D$ such that $|x-y| \geq r_0$, we have*

$$G_{m,n}(x,y) \preceq \frac{\delta(x)\delta(y)}{r_0^{n+2-2m}}. \tag{2.1}$$

Moreover, on D^2 the following estimates hold

$$\delta(x)\delta(y) \preceq G_{m,n}(x,y) \preceq \begin{cases} \frac{\delta(x)\wedge\delta(y)}{|x-y|^{n+1-2m}}, & \text{for } n \geq 2m \\ \delta(x)\wedge\delta(y), & \text{for } n \leq 2m-1. \end{cases}$$

3. THE KATO CLASS $K_{m,n}(D)$

To give new examples of functions belonging to this class we need the following lemma

Lemma 3.1. *For $\lambda,\mu \in \mathbb{R}$ and $x \in D$, let $\rho_{\lambda,\mu}(x) = \frac{1}{\delta^{\lambda}(x)\left[\operatorname{Log}(\frac{2d}{\delta(x)})\right]^{\mu}}$. Then*

$$\rho_{\lambda,\mu} \in L^1(D) \text{ if and only if } \quad \lambda < 1 \text{ or } (\lambda = 1 \text{ and } \mu > 1).$$

Proof. Since for $\lambda < 0$ the function $\rho_{\lambda,\mu}$ is continuous and bounded in D we need only to prove the result for $\lambda \geq 0$.

Since D is a bounded $C^{1,1}$ domain and the function $t \mapsto \frac{1}{t^{\lambda}\left[\operatorname{Log}(\frac{2d}{t})\right]^{\mu}}$ is decreasing near 0 for $\lambda > 0$, then the proof of the lemma on page 726 in [10] can be adapted. □

Proposition 3.2. *Let $m \geq 2$ and $p \in [1,\infty]$. Then $\rho_{\lambda,\mu}(\cdot)L^p(D) \subset K_{m,n}(D)$, provided that:*

i) *For* $n \geq 2m-1$, *we have* $\lambda < 2 + \frac{2(m-1)}{n} - \frac{1}{p}$ *and* $\frac{n}{2(m-1)} < p$.
ii) *For* $n = 2m-2$, *we have* $\lambda < 2 + \frac{n-1}{n} - \frac{1}{p}$ *and* $\frac{n}{n-1} < p$.
iii) *For* $n < 2m-2$, *we have* $\lambda < 3 - \frac{1}{p}$.

Proof. Let $h \in L^p(D)$ and $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in D$ and $\alpha \in (0,1)$, we put

$$I = I(x,\alpha) := \int_{B(x,\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x,y)\rho_{\lambda,\mu}(y)h(y)\,dy.$$

Taking account of Theorem 1.1, we will discuss the following cases:

Case 1. $n \geq 2m-1$. In this case we have

$$I \preceq \int_{B(x,\alpha)\cap D} \frac{h(y)}{|x-y|^{n-2(m-1)}} \frac{dy}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{\mu}}.$$

It follows from the Hölder inequality that

$$I \preceq \|h\|_p \left[\int_{B(x,\alpha)\cap D} \frac{1}{|x-y|^{(n-2(m-1))q}} \frac{dy}{\delta(y)^{(\lambda-2)q}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{q\mu}}\right]^{\frac{1}{q}}.$$

Since $\lambda < 2 + \frac{2(m-1)}{n} - \frac{1}{p}$ and $\frac{n}{2(m-1)} < p$, then $\lambda - 2 < \frac{1}{q} - \frac{n-2(m-1)}{n}$ and $q < \frac{n}{n-2(m-1)}$. Hence we can choose $q' > \max\left(1, \frac{1}{1-(\lambda-2)q}\right)$ so that $q\,q' < \frac{n}{n-2(m-1)}$ and $(\lambda-2)q < 1 - \frac{1}{q'} := \frac{1}{r}$.

We apply the Hölder inequality again and Lemma 3.1 to deduce that

$$I \preceq \|h\|_p \left[\int_D \frac{dy}{\delta(y)^{(\lambda-2)qr}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{qr\mu}}\right]^{\frac{1}{qr}} \alpha^{n-(n-2m+2)qq'}.$$

Hence $\sup_{x\in D} I(x,\alpha) \to 0$ as $\alpha \to 0$.

Case 2. $n = 2m-2$. Assume that $\lambda < 2 + \frac{n-1}{n} - \frac{1}{p}$ and $\frac{n}{n-1} < p$, then $\lambda - 2 < \frac{1}{q} - \frac{1}{n}$ and $q < n$.

Using (1.9), (1.6) and the Hölder inequality we obtain

$$\begin{aligned} I &\preceq \int_{B(x,\alpha)\cap D} \left(1 + \frac{1}{[x,y]}\right) \frac{h(y)}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{\mu}} dy \\ &\preceq \int_{B(x,\alpha)\cap D} \frac{1}{|x-y|} \frac{h(y)}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{\mu}} dy \\ &\preceq \|h\|_p \left[\int_{B(x,\alpha)\cap D} \frac{1}{|x-y|^q} \frac{1}{\delta(y)^{(\lambda-2)q}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{q\mu}} dy\right]^{\frac{1}{q}}. \end{aligned}$$

Let us choose $q' > 1$ and $r = \frac{q'}{q'-1}$ such that $qq' < n$ and $(\lambda-2)qr < 1$. Then, using the Hölder inequality again and Lemma 3.1 we obtain

$$I \preceq \|h\|_p \left[\int_D \frac{dy}{\delta(y)^{(\lambda-2)qr}\left[\mathrm{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{qr\mu}}\right]^{\frac{1}{qr}} \alpha^{n-qq'}.$$

Hence $\sup_{x\in D} I(x,\alpha) \to 0$ as $\alpha \to 0$.

Case 3. $n < 2m-2$. Using Theorem 1.1 and the Hölder inequality we obtain

$$\begin{aligned} I &\preceq \int_{B(x,\alpha)\cap D} \frac{h(y)}{\delta(y)^{\lambda-2}\left[\operatorname{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{\mu}} dy \\ &\preceq \|h\|_p \left[\int_{B(x,\alpha)\cap D} \frac{1}{\delta(y)^{(\lambda-2)q}\left[\operatorname{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{q\mu}} dy\right]^{\frac{1}{q}}. \end{aligned}$$

As in the preceding cases we choose $q' > 1$ so that $(\lambda-2)qq' < 1$ to deduce from the Hölder inequality and Lemma 3.1 that $\sup_{x\in D} I(x,\alpha) \to 0$ as $\alpha \to 0$.

This completes the proof of the proposition. □

Next, we will prove Proposition 1.2. So we need the following results

Lemma 3.3 (see [5]). *Let $x, y \in D$. Then the following properties are satisfied:*

1) *If $\delta(x)\delta(y) \le |x-y|^2$ then*

$$\max(\delta(x)\,,\,\delta(y)) \le \frac{1+\sqrt{5}}{2}\,|x-y|.$$

2) *If $|x-y|^2 \le \delta(x)\delta(y)$ then*

$$\frac{(3-\sqrt{5})}{2}\,\delta(x) \le \delta(y) \le \frac{(3+\sqrt{5})}{2}\delta(x).$$

Lemma 3.4. *Let $q \in K_{m,n}(D)$. Then the function : $x \to \delta^2(x)q(x)$ is in $L^1(D)$.*

Proof. Let $q \in K_{m,n}(D)$. Then by (1.4), there exists $\alpha > 0$ such that for all $x \in D$ we have

$$\int_{(|x-y|\le\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x,y)|q(y)|\,dy \le 1.$$

Let $x_1\,, x_2\,, \dots, x_p \in D$ such that $D \subset \bigcup_{i=1}^p B(x_i,\alpha)$. Then by Corollary 2.2, there exists $C > 0$ such that $\forall\, y \in B(x_i,\alpha)\cap D$ we have

$$\delta^2(y) \le C\frac{\delta(y)}{\delta(x)} G_{m,n}(x_i,y).$$

Hence

$$\begin{aligned} \int_D \delta^2(y)|q(y)|dy &\le C\sum_{i=1}^{p} \int_{B(x_i,\alpha)\cap D} \frac{\delta(y)}{\delta(x_i)} G_{m,n}(x_i,y)|q(y)|\,dy \\ &\le C\,p < \infty. \end{aligned}$$

□

Proof of Proposition 1.2. It follows from Lemmas 3.1 and 3.4 that a necessary condition for $\rho_{\lambda\,,\,\mu}$ to belong to $K_{m,n}(B)$ is that $\lambda < 3$ or $(\lambda = 3$ and $\mu > 1)$. Let us prove that this condition is sufficient.

For $\lambda \le 2$ the results follow from Proposition 3.2 by taking $p = \infty$. Hence we need only to prove the results for $2 < \lambda < 3$ or $(\lambda = 3$ and $\mu > 1)$.

For $x \in D$ and $\alpha \in \left(0, 4e^{-\frac{\mu}{\lambda}}\right)$, we put

$$\begin{aligned} I = I(x,\alpha) &:= \int_{B(x,\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x,y)\rho_{\lambda\,,\,\mu}(y)\,dy \\ &= \int_{B(x,\alpha)\cap D} \frac{G_{m,n}(x,y)}{\delta(x)\delta(y)^{\lambda-1}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\,dy\,. \end{aligned}$$

Taking account of Theorem 1.1 we distinguish the following cases.

Case 1. $n \geq 2m-1$. Then we have

$$\begin{aligned} I &\preceq \int_{B(x,\alpha)\cap D_1} \frac{1}{|x-y|^{n-2(m-1)}} \frac{1}{(\delta(y))^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\,dy \\ &\quad + \int_{B(x,\alpha)\cap D_2} \frac{1}{|x-y|^{n-2(m-1)}} \frac{1}{(\delta(y))^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\,dy \\ &= I_1 + I_2\,, \end{aligned}$$

where

$$D_1 = \{x \in D : |x-y|^2 \leq \delta(x)\delta(y)\} \quad \text{and} \quad D_2 = \{x \in D : \delta(x)\delta(y) \leq |x-y|^2\}.$$

- If $y \in D_1$, then from Lemma 3.3, we have $\delta(x) \sim \delta(y)$ and so $|x-y| \preceq \delta(y)$. Hence

$$\begin{aligned} I_1 &\preceq \int_{B(x,\alpha)} \frac{1}{|x-y|^{n-2m+\lambda}\left[\mathrm{Log}\left(\frac{C}{|x-y|}\right)\right]^{\mu}}\,dy \\ &\preceq \int_0^{\alpha} \frac{r^{2m-(\lambda+1)}}{\left[\mathrm{Log}\left(\frac{C}{r}\right)\right]^{\mu}}\,dr, \end{aligned}$$

  which tends to zero as $\alpha \to 0$.
- If $y \in D_2$, then using Lemma 3.3, we have $\max(\delta(x),\delta(y)) \leq \frac{1+\sqrt{5}}{2}|x-y|$. Hence,

$$I_2 \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^{1} \frac{t^{n-1}}{(1-t)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{1-t}\right)\right]^{\mu}} \left(\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x-t\omega|^{n-2(m-1)}}\right) dt\,,$$

  where σ is the normalized measure on the unit sphere S^{n-1} of $\mathbb{R}^n$.

  Now by elementary calculus, we have

$$\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x-t\omega|^{n-2(m-1)}} \preceq \frac{1}{(|x| \vee t)^{n-2(m-1)}} \preceq t^{2(m-1)-n}.$$

  So

$$I_2 \preceq \int_{1-\alpha\left(\frac{1+\sqrt{5}}{2}\right)}^{1} \frac{t^{2m-3}}{(1-t)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{1-t}\right)\right]^{\mu}} dt,$$

  which tends to zero as α tends to zero.

Case 2. $n = 2m - 2$. In this case we have

$$I \preceq \int_{B(x,\alpha)\cap D_1} \mathrm{Log}\left(2 + \frac{1}{[x,y]^2}\right) \frac{1}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy$$
$$+ \int_{B(x,\alpha)\cap D_2} \mathrm{Log}\left(2 + \frac{1}{[x,y]^2}\right) \frac{1}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy$$
$$= I_1 + I_2.$$

- If $y \in D_1$, it follows from the fact that $\mathrm{Log}(2+t) \leq \sqrt{t}$ for $t \geq 2$ that

$$I_1 \preceq \int_{B(x,\alpha)\cap D_1} \frac{1}{|x-y|(\delta(y))^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy$$
$$\preceq \int_{B(x,\alpha)\cap D_1} \frac{1}{|x-y|^{\lambda-1}\left[\mathrm{Log}\left(\frac{4}{|x-y|}\right)\right]^{\mu}}\, dy$$
$$\preceq \int_0^{\alpha} \frac{r^{n-\lambda}}{\left[\mathrm{Log}\left(\frac{4}{r}\right)\right]^{\mu}}\, dr,$$

which tends to zero as α tends to zero.

- If $y \in D_2$, then

$$I_2 \preceq \int_{B(x,\alpha)\cap D_2} \mathrm{Log}\left(2 + \frac{1}{|x-y|^2}\right) \frac{1}{\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy$$
$$\preceq \int_{B(x,\alpha)\cap D_2} \frac{1}{|x-y|^2\delta(y)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy$$
$$\preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^{1} \frac{t^{n-1}}{(1-t)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{1-t}\right)\right]^{\mu}} \left(\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x-t\omega|^2}\right) dt$$
$$\preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^{1} \frac{t^{n-1}}{(1-t)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{1-t}\right)\right]^{\mu}} \frac{1}{(|x| \vee t)^2}\, dt$$
$$\preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^{1} \frac{t^{n-3}}{(1-t)^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{1-t}\right)\right]^{\mu}}\, dt,$$

which tends to zero as α tends to zero.

Case 3. $n < 2m - 2$. In this case

$$I \preceq \int_{B(x,\alpha)\cap D} \frac{1}{(\delta(y))^{\lambda-2}\left[\mathrm{Log}\left(\frac{4}{\delta(y)}\right)\right]^{\mu}}\, dy.$$

Hence the result follows from Lemma 3.3, using similar arguments as in the above cases.

□

In the sequel we aim at proving Theorem 1.3. Below we present some preliminary results which we will need later.

Proposition 3.5.

$a)$ *For each* $t > 0$ *and all* $x, y \in D$*, we have*

$$\int_0^t s^{m-1} p(s,x,y)\,ds \preceq G_{m,n}(x,y).$$

$b)$ *Let* $0 < t \leq 1$ *and* $x, y \in D$*. Then*

$$G_{m,n}(x,y) \preceq \int_0^t s^{m-1} p(s,x,y)\,ds\,,$$

provided that

i) $n > 2m$ *and* $|x-y| \leq \sqrt{t}$*; or*
ii) $n = 2m$ *and* $[x,y]^2 \leq t$ *; or*
iii) $n = 2m-1$ *and* $|x-y|^2 + 2\delta(x)\delta(y) \leq t$*.*

Proof.
$a)$ Follows from (1.3).
$b)$ We deduce from (1.11) and (1.12) that

$$\int_0^t s^{m-1} p(s,x,y)\,ds \sim |x-y|^{2m-n} \int_{\frac{|x-y|^2}{t}}^{\infty} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}\, r, 1\right) r^{\frac{n}{2}-m-1}\, e^{-r} dr.$$

Next, we distinguish the following cases

i) $n > 2m$. In this case the result follows from (1.7) and Theorem 1.1.
ii) $n = 2m$. Using (1.5) we have

$$\begin{aligned}\int_0^t s^{m-1} p(s,x,y)\,ds &\geq C \int_{\frac{|x-y|^2}{t}}^{2} \frac{\delta(x)\delta(y)}{\delta(x)\delta(y) r + |x-y|^2}\, dr \\ &\geq C\ \mathrm{Log}\left(\frac{[x,y]^2 + \delta(x)\delta(y)}{\delta(x)\delta(y) + t} \cdot \frac{t}{|x-y|^2}\right).\end{aligned}$$

Now since $[x,y]^2 \leq t$ and the function $t \mapsto \frac{t}{\delta(x)\delta(y)+t}$ is nondecreasing, then the result follows from Theorem 1.1.

iii) $n = 2m-1$. As in the proof of Theorem 1.1 we distinguish two cases
- If $\delta(x)\delta(y) \leq |x-y|^2$. In this case the result follows from (1.7).
- If $\delta(x)\delta(y) > |x-y|^2$. Then

$$\int_0^t s^{m-1} p(s,x,y)\,ds \geq C\, \frac{\delta(x)\delta(y)}{|x-y|} \int_{\frac{|x-y|^2}{t}}^{\frac{|x-y|^2}{\delta(x)\delta(y)}} r^{-\frac{1}{2}}\, dr.$$

Since $|x-y|^2 + 2\delta(x)\delta(y) \leq t$, then

$$\left(\left(1 - \frac{1}{\sqrt{2}}\right) \frac{|x-y|}{\sqrt{\delta(x)\delta(y)}} + \frac{|x-y|}{\sqrt{t}}\right)^2 \leq \frac{|x-y|^2}{\delta(x)\delta(y)}.$$

Hence

$$\begin{aligned}\int_0^t s^{m-1} p(s,x,y)\,ds &\geq C \frac{\delta(x)\delta(y)}{|x-y|} \frac{|x-y|}{\sqrt{\delta(x)\delta(y)}} \\ &= C \sqrt{\delta(x)\delta(y)} \\ &\geq C\, G_{m,n}(x,y).\end{aligned}$$

□

Proposition 3.6. *Let $q \in K_{m,n}(D)$. Then for each fixed $\alpha > 0$, we have*

$$\sup_{t\le 1}\left(\sup_{x\in D}\int_{(|x-y|>\alpha)\cap D}\frac{\delta(y)}{\delta(x)}p\left(t,x,y\right)|q(y)|\,dy\right) := M(\alpha) < \infty. \tag{3.1}$$

Proof. Let $0<t<1$, $q\in K_{m,n}(D)$ and $0<\alpha<1$. Then using (1.11) and (1.12) we have

$$\begin{aligned}\int_{(|x-y|>\alpha)\cap D}\frac{\delta(y)}{\delta(x)}p\left(t,x,y\right)|q(y)|\,dy &\preceq \frac{1}{t^{\frac{n}{2}+1}}\int_{(|x-y|>\alpha)\cap D}\delta^2(y)e^{-\frac{|x-y|^2}{t}}|q(y)|\,dy\\ &\preceq \frac{e^{-\frac{\alpha^2}{t}}}{t^{\frac{n}{2}+1}}\int_D\delta^2(y)|q(y)|\,dy.\end{aligned}$$

Hence the result follows from Lemma 3.4. □

Proof of Theorem 1.3. $2) \Rightarrow 1)$ Assume that

$$\lim_{t\to 0}\left(\sup_{x\in D}\int_D\int_0^t\frac{\delta(y)}{\delta(x)}\,s^{m-1}p(s,x,y)|q(y)|\,ds\,dy\right)=0.$$

Then by Proposition 3.5, there exists $C>0$ such that for $\alpha>0$ we have

$$\int_{(|x-y|\le\alpha)\cap D}\frac{\delta(y)}{\delta(x)}G_{m,n}(x,y)|q(y)|\,dy \le C\int_D\int_0^{\alpha^2}\frac{\delta(y)}{\delta(x)}s^{m-1}p(s,x,y)\,dsdy,$$

which shows that q satisfies (1.4).

$1) \Rightarrow 2)$ Suppose that $q\in K_{m,n}(D)$ and let $\varepsilon>0$. Then there exists $0<\alpha<1$ such that

$$\sup_{x\in D}\int_{(|x-y|\le\alpha)\cap D}\frac{\delta(y)}{\delta(x)}G_{m,n}(x,y)|q(y)|\,dy\le\varepsilon.$$

On the other hand, using Proposition 3.5 and (3.1), we have for $0<t<1$

$$\begin{aligned}&\int_D\int_0^t\frac{\delta(y)}{\delta(x)}s^{m-1}p(s,x,y)|q(y)|\,dsdy\\ &\preceq\int_{(|x-y|\le\alpha)\cap D}\int_0^t\frac{\delta(y)}{\delta(x)}s^{m-1}p(s,x,y)|q(y)|\,dsdy\\ &\qquad+\int_{(|x-y|>\alpha)\cap D}\int_0^t\frac{\delta(y)}{\delta(x)}s^{m-1}p(s,x,y)|q(y)|\,dsdy\\ &\preceq\int_{(|x-y|\le\alpha)\cap D}\frac{\delta(y)}{\delta(x)}G_{m,n}(x,y)|q(y)|\,dy\\ &\qquad+\int_0^t\int_{(|x-y|>\alpha)\cap D}\frac{\delta(y)}{\delta(x)}p(s,x,y)|q(y)|\,dsdy\\ &\preceq\varepsilon+t\,M(\alpha).\end{aligned}$$

This achieves the proof. □

Next we assume that $m=1$ and we will give another characterization of the class $K_{1,n}(D)$.

Corollary 3.7. *Let $n\ge 3$ and q be a measurable function. For $\alpha>0$, put*

$$G_\alpha q(x)=\int_D\int_0^\infty e^{-\alpha\,s}\frac{\delta(y)}{\delta(x)}p(s,x,y)|q(y)|\,dsdy,\quad \textit{for } x\in D$$

and

$$a(\alpha) = \sup_{x\in D} \int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s,x,y)|q(y)|\, dyds.$$

Then there exists $C > 0$ *such that*

$$\frac{1}{e}\, a(\alpha) \le \|G_\alpha q\|_\infty \le C\, a(\alpha)\,,$$

where $\|G_\alpha q\|_\infty = \sup_{x\in D} |G_\alpha q(x)|$.

In particular, we have

$$q \in K_{1,n}(D) \Longleftrightarrow \lim_{\alpha\to\infty} \|G_\alpha q\|_\infty = 0.$$

Proof. Let $\alpha > 0$. Then using the Fubini theorem, we obtain for $x \in D$

$$\begin{aligned} G_\alpha q(x) &= \int_0^\infty \alpha e^{-\alpha t} \left[\int_0^t \int_D \frac{\delta(y)}{\delta(x)} p(s,x,y)|q(y)|\, dyds\right] dt \\ &= \int_0^\infty e^{-t} \left[\int_0^{\frac{t}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s,x,y)|q(y)|\, dyds\right] dt. \end{aligned}$$

Hence, $\frac{1}{e}\, a(\alpha) \le \|G_\alpha q\|_\infty$.

On the other hand if we denote by $[t]$ the integer part of t, then we have

$$\begin{aligned} G_\alpha q(x) &\le \int_0^\infty e^{-t} \left[\sum_{k=0}^{[t]} \int_{\frac{k}{\alpha}}^{\frac{k+1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s,x,y)|q(y)|\, dyds\right] dt \\ &\le \int_0^\infty e^{-t} \left[\sum_{k=0}^{[t]} \int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p\left(s+\frac{k}{\alpha},x,y\right) |q(y)|\, dyds\right] dt. \end{aligned}$$

Now, using the Chapmann-Kolmogorov identity and the Fubini theorem we obtain

$$\begin{aligned} &\int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)}\, p\left(s+\frac{k}{\alpha},x,y\right) |q(y)| dyds \\ &= \int_D \left(\int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(z)} p(s,z,y)|q(y)| dyds\right) \frac{\delta(z)}{\delta(x)} p\left(\frac{k}{\alpha},x,z\right) dz \\ &\le a(\alpha) \int_D \frac{\delta(z)}{\delta(x)} p\left(\frac{k}{\alpha},x,z\right) dz. \end{aligned}$$

Since the first eigenfunction φ_1 associated to $-\Delta$ satisfies $\varphi_1(x) \sim \delta(x)$ and

$$\int_D p(t,x,z)\varphi_1(z)dz = e^{-\lambda_1 t}\varphi_1(x) \le \varphi_1(x),$$

then

$$\|G_\alpha q\|_\infty \le Ca(\alpha).$$

So, the last assertion follows from Theorem 1.3. □

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