

BOUNDS FOR SOME PERTURBED ČEBYŠEV FUNCTIONALS

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Abstract: Bounds for the perturbed Čebyšev functionals $C(f, g) - \mu C(e, g)$ and $C(f, g) - \mu C(e, g) - \nu C(f, e)$ when $\mu, \nu \in \mathbb{R}$ and e is the identity function on the interval $[a, b]$, are given. Applications for some Grüss' type inequalities are also provided.



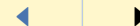
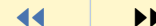
**Perturbed Čebyšev
Functionals**

S.S. Dragomir

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1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$(1.1) \quad C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt.$$

In 1934, Grüss [5] showed that

$$(1.2) \quad |C(f, g)| \leq \frac{1}{4} (M - m) (N - n),$$

provided that there exists the real numbers m, M, n, N such that

$$(1.3) \quad m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$(1.4) \quad |C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2,$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)|$. The constant $\frac{1}{12}$ can be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$ while $\|f'\|_\infty = \text{ess sup}_{t \in [a, b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [9]:

$$(1.5) \quad |C(f, g)| \leq \frac{1}{8} (b-a) (M - m) \|g'\|_\infty,$$



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provided that f is Lebesgue integrable and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty [a, b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of euclidean norms of the derivative was considered by A. Lupaş in [7] in which he proved that

$$(1.6) \quad |C(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b - a),$$

provided that f, g are absolutely continuous and $f', g' \in L_2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [1] have proved the following results:

$$(1.7) \quad |C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b - a} \left(\int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$(1.8) \quad |C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 \cdot \frac{1}{b - a} \operatorname{ess\,sup}_{t \in [a, b]} \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right|,$$

provided that $f \in L_p [a, b]$ and $g \in L_q [a, b]$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1$; $p = 1, q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.7) we obtain

$$(1.9) \quad |C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right| dt \\ \leq \|g\|_\infty \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right| dt$$

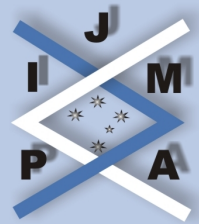
and if g satisfies (1.3), then

$$\begin{aligned}
 (1.10) \quad |C(f, g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_{\infty} \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\
 &\leq \left\| g - \frac{n+N}{2} \right\|_{\infty} \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\
 &\leq \frac{1}{2} (N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.
 \end{aligned}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [4]. However, the sharpness of the constant $\frac{1}{2}$, a generalisation for the abstract Lebesgue integral and the discrete version of it have been obtained in [2].

For other recent results on the Grüss inequality, see [6], [8] and [10] and the references therein.

The aim of the present paper is to establish Grüss type inequalities for some perturbed Čebyšev functionals. For this purpose, two integral representations of the functionals $C(f, g) - \mu C(e, g)$ and $C(f, g) - \mu C(e, g) - \nu C(f, e)$ when $\mu, \nu \in \mathbb{R}$ and $e(t) = t$, $t \in [a, b]$ are given.



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2. Representation Results

The following representation result can be stated.

Lemma 2.1. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and g is Lebesgue integrable on $[a, b]$, then*

$$(2.1) \quad C(f, g) = \frac{1}{(b-a)^2} \int_a^b \int_a^b Q(t, s) [g(s) - \lambda] f'(t) ds dt$$

for any $\lambda \in \mathbb{R}$, where the kernel $Q : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$(2.2) \quad Q(t, s) := \begin{cases} t - b & \text{if } a \leq s \leq t \leq b, \\ t - a & \text{if } a \leq t < s \leq b. \end{cases}$$

Proof. We observe that for $\lambda \in \mathbb{R}$ we have $C(f, \lambda) = 0$ and thus it suffices to prove (2.1) for $\lambda = 0$.

By Fubini's theorem, we have

$$(2.3) \quad \int_a^b \int_a^b Q(t, s) g(s) f'(t) ds dt = \int_a^b \left(\int_a^b Q(t, s) f'(t) dt \right) g(s) ds.$$

By the definition of $Q(t, s)$ and integrating by parts, we have successively,

$$(2.4) \quad \begin{aligned} \int_a^b Q(t, s) f'(t) dt &= \int_a^s Q(t, s) f'(t) dt + \int_s^b Q(t, s) f'(t) dt \\ &= \int_a^s (t - a) f'(t) dt + \int_s^b (t - b) f'(t) dt \end{aligned}$$

$$\begin{aligned}
&= (s - a) f(s) - \int_a^s f(t) dt + (b - s) f(s) - \int_s^b f(t) dt \\
&= (b - a) f(s) - \int_a^b f(t) dt,
\end{aligned}$$

for any $s \in [a, b]$.

Now, integrating (2.4) multiplied with $g(s)$ over $s \in [a, b]$, we deduce

$$\begin{aligned}
\int_a^b \left(\int_a^b Q(t, s) f'(t) dt \right) g(s) ds &= \int_a^b \left[(b - a) f(s) - \int_a^b f(t) dt \right] g(s) ds \\
&= (b - a) \int_a^b f(s) g(s) ds - \int_a^b f(s) ds \cdot \int_a^b g(s) ds \\
&= (b - a)^2 C(f, g)
\end{aligned}$$

and the identity is proved. \square

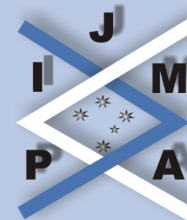
Utilising the linearity property of $C(\cdot, \cdot)$ in each argument, we can state the following equality:

Theorem 2.2. *If $e : [a, b] \rightarrow \mathbb{R}$, $e(t) = t$, then under the assumptions of Lemma 2.1 we have:*

$$(2.5) \quad C(f, g) = \mu C(e, g) + \frac{1}{(b - a)^2} \int_a^b \int_a^b Q(t, s) [g(s) - \lambda] [f'(t) - \mu] dt ds$$

for any $\lambda, \mu \in \mathbb{R}$, where

$$(2.6) \quad C(e, g) = \frac{1}{b - a} \int_a^b t g(t) dt - \frac{a + b}{2} \int_a^b g(t) dt.$$



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The second representation result is incorporated in

Lemma 2.3. *If $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$, then*

$$(2.7) \quad C(f, g) = \frac{1}{(b-a)^2} \int_a^b \int_a^b K(t, s) f'(t) g'(s) dt ds,$$

where the kernel $K : [a, b] \rightarrow \mathbb{R}$ is defined by

$$(2.8) \quad K(t, s) := \begin{cases} (b-t)(s-a) & \text{if } a \leq s \leq t \leq b, \\ (t-a)(b-s) & \text{if } a \leq t < s \leq b. \end{cases}$$

Proof. By Fubini's theorem we have

$$(2.9) \quad \int_a^b \int_a^b K(t, s) f'(t) g'(s) dt ds = \int_a^b \left(\int_a^b K(t, s) g'(s) ds \right) f'(t) dt.$$

By the definition of K and integrating by parts, we have successively:

$$(2.10) \quad \begin{aligned} & \int_a^b K(t, s) g'(s) ds \\ &= \int_a^t K(t, s) g'(s) ds + \int_t^b K(t, s) g'(s) ds \\ &= (b-t) \int_a^t (s-a) g'(s) ds + (t-a) \int_t^b (b-s) g'(s) ds \\ &= (b-t) \left[(t-a) g(t) - \int_a^t g(s) ds \right] \\ & \quad + (t-a) \left[-(b-t) g(t) + \int_t^b g(s) ds \right] \end{aligned}$$

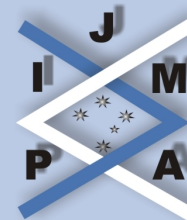
$$= (t - a) \int_t^b g(s) ds - (b - t) \int_a^t g(s) ds,$$

for any $t \in [a, b]$.

Multiplying (2.10) by $f'(t)$ and integrating over $t \in [a, b]$, we have:

$$\begin{aligned}
 (2.11) \quad & \int_a^b \left(\int_a^b K(t, s) g'(s) ds \right) f'(t) dt \\
 &= \int_a^b \left[(t - a) \int_t^b g(s) ds - (b - t) \int_a^t g(s) ds \right] f'(t) dt \\
 &= f(t) \left[(t - a) \int_t^b g(s) ds - (b - t) \int_a^t g(s) ds \right] \Big|_a^b \\
 &\quad - \int_a^b f(t) \left[(t - a) \int_t^b g(s) ds - (b - t) \int_a^t g(s) ds \right]' dt \\
 &= \int_a^b f(t) \left[\int_t^b g(s) ds - (t - a) g(t) + \int_a^t g(s) ds - (b - t) g(t) \right] \\
 &= - \int_a^b f(t) \left[\int_a^b g(s) ds - (b - a) g(t) \right] dt \\
 &= (b - a) \int_a^b g(t) f(t) dt - \int_a^b f(t) dt \cdot \int_a^b g(t) dt \\
 &= (b - a)^2 C(f, g).
 \end{aligned}$$

By (2.11) and (2.9) we deduce the desired result. \square



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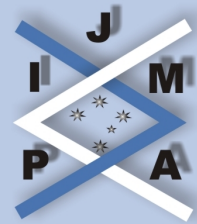
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Theorem 2.4. *With the assumptions of Lemma 2.3, we have for any $\nu, \mu \in \mathbb{R}$ that:*

$$(2.12) \quad C(f, g) = \mu C(e, g) + \nu C(f, e) \\ + \frac{1}{(b-a)^2} \int_a^b \int_a^b K(t, s) [f'(t) - \mu] [g'(s) - \nu] dt ds.$$

Proof. Follows by Lemma 2.3 on observing that $C(e, e) = 0$ and

$$C(f - \mu e, g - \nu e) = C(f, g) - \mu C(e, g) - \nu C(f, e)$$

for any $\mu, \nu \in \mathbb{R}$. □



3. Bounds in Terms of Lebesgue Norms of g and f'

Utilising the representation (2.5) we can state the following result:

Theorem 3.1. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable on $[a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then

$$(3.1) \quad |C(f, g) - \mu C(e, g)| \leq \begin{cases} \frac{1}{3} (b-a) \|f' - \mu\|_{\infty} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_{\infty} & \text{if } f', g \in L_{\infty}[a, b]; \\ \frac{2^{1/q} (b-a)^{\frac{p-q}{pq}}}{[(q+1)(q+2)]^{1/q}} \|f' - \mu\|_p \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_p & \text{if } f', g \in L_p[a, b], \\ (b-a)^{-1} \|f' - \mu\|_1 \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

for any $\mu \in \mathbb{R}$.

Proof. From (2.5), we have

$$(3.2) \quad |C(f, g) - \mu C(e, g)| \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b |Q(t, s)| |g(s) - \lambda| |f'(t) - \mu| dt ds \leq \|g - \lambda\|_{\infty} \|f' - \mu\|_{\infty} \frac{1}{(b-a)^2} \int_a^b \int_a^b |Q(t, s)| dt ds.$$



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However, by the definition of Q we have for $\alpha \geq 1$ that

$$\begin{aligned} I(\alpha) &:= \int_a^b \int_a^b |Q(t, s)|^\alpha dt ds \\ &= \int_a^b \left(\int_a^t |t - b|^\alpha ds + \int_t^b |t - a|^\alpha ds \right) dt \\ &= \int_a^b [(t - a)(b - t)^\alpha + (b - t)(t - a)^\alpha] dt. \end{aligned}$$

Since

$$\int_a^b (t - a)(b - t)^\alpha dt = \frac{(b - a)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)}$$

and

$$\int_a^b (b - t)(t - a)^\alpha dt = \frac{(b - a)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)},$$

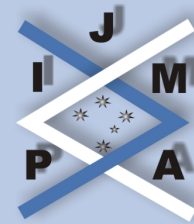
hence

$$I(\alpha) = \frac{2(b - a)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)}, \quad \alpha \geq 1.$$

Then we have

$$\frac{1}{(b - a)^2} \int_a^b \int_a^b |Q(t, s)| dt ds = \frac{b - a}{3},$$

and taking the infimum over $\lambda \in \mathbb{R}$ in (3.2), we deduce the first part of (3.1).



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Utilising the Hölder inequality for double integrals we also have

$$\begin{aligned} & \int_a^b \int_a^b |Q(t, s)| |g(s) - \lambda| |f'(t) - \mu| dt ds \\ & \leq \left(\int_a^b \int_a^b |Q(t, s)|^q dt ds \right)^{\frac{1}{q}} \left(\int_a^b \int_a^b |g(s) - \lambda|^p |f'(t) - \mu|^p dt ds \right)^{\frac{1}{p}} \\ & = \frac{2^{1/q} (b-a)^{1+\frac{2}{q}}}{[(q+1)(q+2)]^{1/q}} \|g - \lambda\|_p \|f' - \mu\|_p, \end{aligned}$$

which provides, by the first inequality in (3.2), the second part of (3.1).

For the last part, we observe that $\sup_{(t,s) \in [a,b]^2} |Q(t, s)| = b - a$ and then

$$\int_a^b \int_a^b |Q(t, s)| |g(s) - \lambda| |f'(t) - \mu| dt ds \leq (b-a) \|g - \lambda\|_1 \|f' - \mu\|_1.$$

This completes the proof. □

Remark 1. The above inequality (3.1) is a source of various inequalities as will be shown in the following.

- For instance, if $-\infty < m \leq g(t) \leq M < \infty$ for a.e. $t \in [a, b]$, then $\|g - \frac{m+M}{2}\|_\infty \leq \frac{1}{2}(M - m)$ and $\|g - \frac{m+M}{2}\|_p \leq \frac{1}{2}(M - m)(b-a)^{1/p}$, $p \geq 1$. Then for any $\mu \in \mathbb{R}$ we have

$$(3.3) \quad |C(f, g) - \mu C(e, g)| \leq \begin{cases} \frac{1}{6} (b-a) (M-m) \|f' - \mu\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{2^{-1/p} (b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}} (M-m) \|f' - \mu\|_p & \text{if } f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (M-m) \|f' - \mu\|_1, & \end{cases}$$



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which gives for $\mu = 0$ that

$$(3.4) \quad |C(f, g)| \leq \begin{cases} \frac{1}{6} (b-a) (M-m) \|f'\|_{\infty} & \text{if } f' \in L_{\infty} [a, b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}} (M-m) \|f'\|_p & \text{if } f' \in L_p [a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (M-m) \|f'\|_1. & \end{cases}$$

2. If $-\infty < \gamma \leq f'(t) \leq \Gamma < \infty$ for a.e. $t \in [a, b]$, then $\|f' - \frac{\gamma+\Gamma}{2}\|_{\infty} \leq \frac{1}{2} |\Gamma - \gamma|$ and $\|f' - \frac{\gamma+\Gamma}{2}\|_p \leq \frac{1}{2} |\Gamma - \gamma| (b-a)^{1/p}$, $p \geq 1$. Then we have from (3.1) that

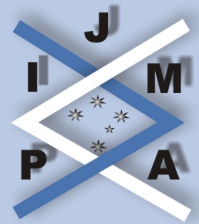
$$(3.5) \quad \left| C(f, g) - \frac{\gamma + \Gamma}{2} C(e, g) \right| \leq \begin{cases} \frac{1}{6} (b-a) (\Gamma - \gamma) \inf_{\xi \in \mathbb{R}} \|g - \xi\|_{\infty} & \text{if } g \in L_{\infty} [a, b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}} (\Gamma - \gamma) \inf_{\xi \in \mathbb{R}} \|g - \xi\|_p & \text{if } g \in L_p [a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (\Gamma - \gamma) \inf_{\xi \in \mathbb{R}} \|g - \xi\|_1. & \end{cases}$$

Moreover, if we also assume that $-\infty < m \leq g(t) \leq M < \infty$ for a.e. $t \in [a, b]$, then by (3.5) we also deduce:

$$(3.6) \quad \left| C(f, g) - \frac{\gamma + \Gamma}{2} C(e, g) \right|$$

$$\leq \begin{cases} \frac{1}{12} (b - a) (\Gamma - \gamma) (M - m) \\ \frac{2^{1-1/p}(b-a)}{[(q+1)(q+2)]^{1/q}} (\Gamma - \gamma) (M - m) & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} (\Gamma - \gamma) (M - m) (b - a). \end{cases}$$

Observe that the first inequality in (3.6) is better than the others.



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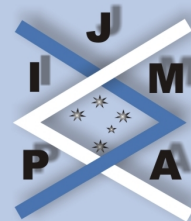
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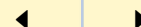
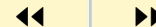
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4. Bounds in Terms of Lebesgue Norms of f' and g'

We have the following result:

Theorem 4.1. Assume that $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$, then

$$(4.1) \quad |C(f, g) - \mu C(e, g) - \nu C(f, e)| \leq \begin{cases} \frac{1}{12} (b-a)^2 \|f' - \mu\|_{\infty} \|g' - \nu\|_{\infty} & \text{if } f', g' \in L_{\infty}[a, b]; \\ \left[\frac{B(q+1, q+1)}{q+1} \right]^{\frac{1}{q}} (b-a)^{2/q} \|f' - \mu\|_p \|g' - \nu\|_p & \text{if } f', g' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} \|f' - \mu\|_1 \|g' - \nu\|_1; & \end{cases}$$

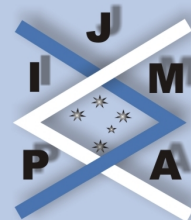
for any $\mu, \nu \in \mathbb{R}$.

Proof. From (2.12), we have

$$(4.2) \quad |C(f, g) - \mu C(e, g) - \nu C(f, e)| \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t, s)| |f'(t) - \mu| |g'(s) - \nu| dt ds.$$

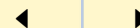
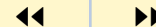
Define

$$(4.3) \quad \begin{aligned} J(\alpha) &:= \int_a^b \int_a^b |K(t, s)|^{\alpha} dt ds \\ &= \int_a^b \left[\int_a^t (b-t)^{\alpha} (s-a)^{\alpha} ds + \int_t^b (t-a)^{\alpha} (b-s)^{\alpha} ds \right] dt \\ &= \frac{1}{\alpha+1} \left[\int_a^b (b-t)^{\alpha} (t-a)^{\alpha+1} dt + \int_a^b (t-a)^{\alpha} (b-t)^{\alpha+1} dt \right]. \end{aligned}$$



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Since

$$\begin{aligned}\int_a^b (t-a)^p (b-t)^q dt &= (b-a)^{p+q+1} \int_0^1 s^p (1-s)^q ds \\ &= (b-a)^{p+q+1} B(p+1, q+1),\end{aligned}$$

hence, by (4.3),

$$J(\alpha) = \frac{2(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1, \alpha+2), \quad \alpha \geq 1.$$

As it is well known that

$$B(p, q+1) = \frac{q}{p+q} B(p, q),$$

then for $p = \alpha + 1$, $q = \alpha + 1$ we have $B(\alpha + 1, \alpha + 2) = \frac{1}{2} B(\alpha + 1, \alpha + 1)$.

Then we have

$$J(\alpha) = \frac{(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1, \alpha+1), \quad \alpha \geq 1.$$

Taking into account that

$$\begin{aligned}\frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t, s)| |f'(t) - \mu| |g'(s) - \nu| dt ds \\ \leq \|f' - \mu\|_\infty \|g' - \nu\|_\infty \frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t, s)| dt ds \\ = \|f' - \mu\|_\infty \|g' - \nu\|_\infty (b-a)^2 B(2, 3) \\ = \frac{1}{12} (b-a)^2 \|f' - \mu\|_\infty \|g' - \nu\|_\infty,\end{aligned}$$



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we deduce from (4.2) the first part of (4.1).

By the Hölder integral inequality for double integrals, we have

$$\begin{aligned}(4.4) \quad & \int_a^b \int_a^b |K(t, s)| |f'(t) - \mu| |g'(s) - \nu| dt ds \\ & \leq \left(\int_a^b \int_a^b |K(t, s)|^q dt ds \right)^{\frac{1}{q}} \|f' - \mu\|_p \|g' - \nu\|_p \\ & = \left[\frac{(b-a)^{2q+2}}{q+1} B(q+1, q+2) \right]^{\frac{1}{q}} \|f' - \mu\|_p \|g' - \nu\|_p \\ & = (b-a)^{2+2/q} \left[\frac{B(q+1, q+1)}{q+1} \right]^{\frac{1}{q}} \|f' - \mu\|_p \|g' - \nu\|_p.\end{aligned}$$

Utilising (4.2) and (4.4) we deduce the second part of (4.1).

By the definition of $K(t, s)$ we have, for $a \leq s \leq t \leq b$, that

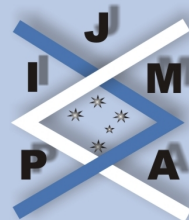
$$K(t, s) = (b-t)(s-a) \leq (b-t)(t-a) \leq \frac{1}{4}(b-a)^2$$

and for $a \leq t < s \leq b$, that

$$K(t, s) = (t-a)(b-s) \leq (t-a)(b-t) \leq \frac{1}{4}(b-a)^2,$$

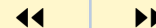
therefore

$$\sup_{(t,s) \in [a,b]} |K(t, s)| = \frac{1}{4}(b-a)^2.$$



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Due to the fact that

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t,s)| |f'(t) - \mu| |g'(s) - \nu| dt ds \\ & \leq \sup_{(t,s) \in [a,b]} |K(t,s)| \frac{1}{(b-a)^2} \int_a^b \int_a^b |f'(t) - \mu| |g'(s) - \nu| dt ds \\ & = \frac{1}{4} \|f' - \mu\|_1 \|g' - \nu\|_1, \end{aligned}$$

then from (4.2) we obtain the last part of (4.1). \square

Remark 2. When $\mu = \nu = 0$, we obtain from (4.1) the following Grüss type inequalities:

$$(4.5) \quad |C(f,g)| \leq \begin{cases} \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty & \text{if } f', g' \in L_\infty[a,b]; \\ \left[\frac{B(q+1, q+1)}{q+1} \right]^{\frac{1}{q}} (b-a)^{2/q} \|f'\|_p \|g'\|_p & \text{if } f', g' \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} \|f'\|_1 \|g'\|_1. & \end{cases}$$

Notice that the first inequality in (4.5) is exactly the Čebyšev inequality for which $\frac{1}{12}$ is the best possible constant.

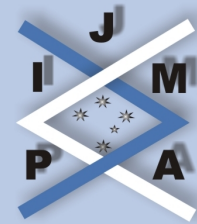
If we assume that there exists $\gamma, \Gamma, \phi, \Phi$ such that $-\infty < \gamma \leq f'(t) \leq \Gamma < \infty$ and $-\infty < \phi \leq g'(t) \leq \Phi < \infty$ for a.e. $t \in [a, b]$, then we deduce from (4.1) the

following inequality

$$(4.6) \quad \left| C(f, g) - \frac{\gamma + \Gamma}{2} \cdot C(e, g) - \frac{\phi + \Phi}{2} \cdot C(f, e) \right| \leq \frac{1}{48} (b - a)^2 (\Gamma - \gamma) (\Phi - \phi).$$

We also observe that the constant $\frac{1}{48}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

The sharpness of the constant follows by the fact that for $\Gamma = -\gamma$, $\Phi = -\phi$ we deduce from (4.6) the Čebyšev inequality which is sharp.



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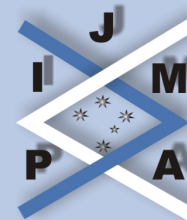
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journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] P. CERONE AND S.S. DRAGOMIR, New bounds for the Čebyšev functional, *App. Math. Lett.*, **18** (2005), 603–611.
- [2] P. CERONE AND S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37–49. Preprint *RGMA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE <http://rgmia.vu.edu.au/v8n2.html>].
- [3] P.L. CHEBYSHEV, Sur les expressions approximatives des intégrals définis par les autres prises entre les même limites, *Proc. Math. Soc. Charkov*, **2** (1882), 93–98.
- [4] X.-L. CHENG AND J. SUN, Note on the perturbed trapezoid inequality, *J. Ineq. Pure & Appl. Math.*, **3**(2) (2002), Art. 29. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=181>].
- [5] G. GRÜSS, Über das Maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.
- [6] X. LI, R.N. MOHAPATRA AND R.S. RODRIGUEZ, Grüss-type inequalities. *J. Math. Anal. Appl.*, **267**(2) (2002), 434–443.
- [7] A. LUPAȘ, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(38)(2) (1973), 219–222.
- [8] A. McD. MERCER, An improvement of the Grüss inequality, *J. Inequal. Pure Appl. Math.*, **6**(4) (2005), Art. 93. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=566>].



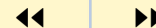
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issn: 1443-5756

- [9] A.M. OSTROWSKI, On an integral inequality, *Aequat. Math.*, **4** (1970), 358–373.
- [10] B.G. PACHPATTE, On Grüss like integral inequalities via Pompeiu’s mean value theorem, *J. Inequal. Pure Appl. Math.*, **6**(3) (2005), Art. 82. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=555>].



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