



ON CERTAIN PROPERTIES OF NEIGHBORHOODS OF MULTIVALENT FUNCTIONS INVOLVING THE GENERALIZED SAITOH OPERATOR

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Abstract: In this paper, we introduce the generalized Saitoh operator $L_p(a, c, \eta)$ and using this operator, the new subclasses $\mathcal{H}_{n,m}^{p,b}(a, c, \eta)$, $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$, $\mathcal{H}_{n,m}^{p,b,\alpha}(a, c, \eta)$ and $\mathcal{L}_{n,m}^{p,b,\alpha}(a, c, \eta; \mu)$ of the class of multivalent functions denoted by $\mathcal{A}_p(n)$ are defined. Further for functions belonging to these classes, certain properties of neighborhoods are studied.

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1. Introduction

Let $\mathcal{A}_p(n)$ be the class of normalized functions f of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad (n, p \in \mathbb{N}),$$

which are analytic and p -valent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{T}_p(n)$ be the subclass of $\mathcal{A}_p(n)$, consisting of functions f of the form

$$(1.2) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (a_k \geq 0, n, p \in \mathbb{N}),$$

which are p -valent in \mathcal{U} .

The Hadamard product of two power series

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k$$

is defined as

$$(f * g)(z) = z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k.$$

Definition 1.1. For $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{\dots, -2, -1, 0\}$ and $\eta \in \mathbb{R}$ ($\eta \geq 0$), the operator $L_p(a, c, \eta) : \mathcal{A}_p(n) \rightarrow \mathcal{A}_p(n)$, is defined as

$$(1.3) \quad L_p(a, c, \eta)f(z) = \phi_p(a, c, z) * D_\eta f(z),$$

where

$$D_\eta f(z) = (1 - \eta)f(z) + \frac{\eta}{p} z f'(z), \quad (\eta \geq 0, z \in \mathcal{U})$$



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and

$$\phi_p(a, c, z) = z^p + \sum_{k=n+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k, \quad z \in \mathcal{U}$$

and $(x)_k$ denotes the Pochhammer symbol given by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0, \\ x(x+1) \cdots (x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, 3, \dots\}. \end{cases}$$

In particular, we have, $L_1(a, c, \eta) \equiv L(a, c, \eta)$.

Further, if $f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k$, then

$$L_p(a, c, \eta)f(z) = z^p + \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k.$$

Remark 1. For $\eta = 0$ and $n = 1$, we obtain the Saitoh operator [7] which yields the Carlson - Shaffer operator [1] for $\eta = 0$ and $n = p = 1$.

For any function $f \in \mathcal{T}_p(n)$ and $\delta \geq 0$, the (n, δ) -neighborhood of f is defined as,

$$(1.4) \quad \mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k |a_k - b_k| \leq \delta \right\}.$$

For the function $h(z) = z^p$, ($p \in \mathbb{N}$) we have,

$$(1.5) \quad \mathcal{N}_{n,\delta}(h) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k |b_k| \leq \delta \right\}.$$

The concept of neighborhoods was first introduced by Goodman [2] and then generalized by Ruscheweyh [6].



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Definition 1.2. A function $f \in \mathcal{T}_p(n)$ is said to be in the class $\mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ if

$$(1.6) \quad \left| \frac{1}{b} \left(\frac{z (L_p(a, c, \eta)f(z))^{(m+1)}}{(L_p(a, c, \eta)f(z))^{(m)}} - (p - m) \right) \right| < 1,$$

where $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $a > 0$, $\eta \geq 0$, $p > m$, $b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{U}$.

Definition 1.3. A function $f \in \mathcal{T}_p(n)$ is said to be in the class $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$ if

$$(1.7) \quad \left| \frac{1}{b} \left[p(1 - \mu) \left(\frac{L_p(a, c, \eta)f(z)}{z} \right)^{(m)} + \mu (L_p(a, c, \eta)f(z))^{(m+1)} - (p - m) \right] \right| < p - m,$$

where $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $a > 0$, $\eta \geq 0$, $p > m$, $\mu \geq 0$, $b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{U}$.



2. Coefficient Bounds

In this section, we determine the coefficient inequalities for functions to be in the subclasses $\mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ and $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$.

Theorem 2.1. *Let $f \in \mathcal{T}_p(n)$. Then, $f \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ if and only if*

$$(2.1) \quad \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (k + |b| - p) a_k \leq |b| \binom{p}{m}.$$

Proof. Let $f \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$. Then, by (1.6) and (1.7) we can write,

$$(2.2) \quad \Re \left\{ \frac{\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (p - k) a_k z^{k-p}}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k z^{k-p}} \right\} > -|b|, \quad (z \in \mathcal{U}).$$

Taking $z = r$, ($0 \leq r < 1$) in (2.2), we see that the expression in the denominator on the left hand side of (2.2), is positive for $r = 0$ and for all r , $0 \leq r < 1$. Hence, by letting $r \mapsto 1^-$ through real values, expression (2.2) yields the desired assertion (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain,

$$\begin{aligned} & \left| \frac{z (L_p(a, c, \eta) f(z))^{(m+1)}}{(L_p(a, c, \eta) f(z))^{(m)}} - (p - m) \right| \\ &= \left| \frac{\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (p - k) a_k z^{k-m}}{\binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k z^{k-m}} \right| \end{aligned}$$

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$$\begin{aligned} &\leq \frac{|b| \left[\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p} \binom{k}{m} a_k}{(c)_{k-p} \binom{k}{m} a_k} \right]}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p} \binom{k}{m} a_k}{(c)_{k-p} \binom{k}{m} a_k}} \\ &= |b|. \end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$. □

On similar lines, we can prove the following theorem.

Theorem 2.2. *A function $f \in \mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$ if and only if*

$$(2.3) \quad \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \eta \right] \frac{(a)_{k-p} \binom{k-1}{m}}{(c)_{k-p} \binom{k-1}{m}} [\mu(k-1) + 1] a_k \leq (p-m) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right].$$



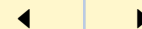
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3. Inclusion Relationships Involving (n, δ) -Neighborhoods

In this section, we prove certain inclusion relationships for functions belonging to the classes $\mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ and $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$.

Theorem 3.1. *If*

$$(3.1) \quad \delta = \frac{(n+p)|b|\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}, \quad (p > |b|),$$

then $\mathcal{H}_{n,m}^{p,b}(a, c, \eta) \subset \mathcal{N}_{n,\delta}(h)$.

Proof. Let $f \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$. By Theorem 2.1, we have,

$$(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty}a_k \leq |b|\binom{p}{m},$$

which implies

$$(3.2) \quad \sum_{k=n+p}^{\infty}a_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}.$$

Using (2.1) and (3.2), we have,

$$\begin{aligned} & \left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty}ka_k \\ & \leq |b|\binom{p}{m} + (p-|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty}a_k \end{aligned}$$



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$$\leq |b| \binom{p}{m} + (p - |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m} \frac{|b| \binom{p}{m}}{(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}$$

$$= |b| \binom{p}{m} \frac{n+p}{n + |b|}.$$

That is,

$$\sum_{k=n+p}^{\infty} k a_k \leq \frac{|b|(n+p) \binom{p}{m}}{(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}} = \delta, \quad (p > |b|).$$

Thus, by the definition given by (1.5), $f \in \mathcal{N}_{n,\delta}(h)$. □

Similarly, we prove the following theorem.

Theorem 3.2. *If*

$$(3.3) \quad \delta = \frac{(p-m)(n+p) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right]}{[\mu(n+p-1) + 1] \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}, \quad (\mu > 1)$$

then $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu) \subset \mathcal{N}_{n,\delta}(h)$.



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4. Further Neighborhood Properties

In this section, we determine the neighborhood properties of functions belonging to the subclasses $\mathcal{H}_{n,m}^{p,b,\alpha}(a, c, \eta)$ and $\mathcal{L}_{n,m}^{p,b,\alpha}(a, c, \eta; \mu)$.

For $0 \leq \alpha < p$ and $z \in \mathcal{U}$, a function f is said to be in the class $\mathcal{H}_{n,m}^{p,b,\alpha}(a, c, \eta)$ if there exists a function $g \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ such that

$$(4.1) \quad \left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha.$$

For $0 \leq \alpha < p$ and $z \in \mathcal{U}$, a function f is said to be in the class $\mathcal{L}_{n,m}^{p,b,\alpha}(a, c, \eta; \mu)$ if there exists a function $g \in \mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$ such that the inequality (4.1) holds true.

Theorem 4.1. *If $g \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ and*

$$(4.2) \quad \alpha = p - \frac{\delta(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}{(n+p) \left[(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m} - |b| \binom{p}{m} \right]},$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{H}_{n,m}^{p,b,\alpha}(a, c, \eta)$.

Proof. Let $f \in \mathcal{N}_{n,\delta}(g)$. Then,

$$(4.3) \quad \sum_{k=n+p}^{\infty} k |a_k - b_k| \leq \delta,$$

which yields the coefficient inequality,

$$(4.4) \quad \sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+p}, \quad (n \in \mathbb{N}).$$



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Since $g \in \mathcal{H}_{n,m}^{p,b}(a, c, \eta)$, by (3.2) we have,

$$(4.5) \quad \sum_{k=n+p}^{\infty} b_k \leq \frac{|b| \binom{p}{m}}{(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}$$

so that,

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{k=n+p}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+p}^{\infty} b_k} \\ &\leq \frac{\delta}{n+p} \frac{(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}{\left[(n + |b|) \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m} - |b| \binom{p}{m} \right]} \\ &= p - \alpha. \end{aligned}$$

Thus, by definition, $f \in \mathcal{H}_{n,m}^{p,b,\alpha}(a, c, \eta)$ for α given by (4.2). □

On similar lines, we prove the following theorem.

Theorem 4.2. *If $g \in \mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$ and*

$$(4.6) \quad \alpha = p - \frac{1}{(n+p)} \times \frac{\delta [\mu(n+p-1) + 1] \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p-1}{m}}{\left[\{\mu(n+p-1) + 1\} \left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p-1}{m} - (p-m) \left(\frac{|b|-1}{m!} + \binom{p}{m} \right) \right]},$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{L}_{n,m}^{p,b,\alpha}(a, c, \eta; \mu)$.

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