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ON THE TRIANGLE INEQUALITY IN QUASI-BANACH SPACES

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ABSTRACT. In this paper, we show the triangle inequality and its reverse inequality in quasi-Banach spaces.

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1. INTRODUCTION

The triangle inequality is one of the most fundamental inequalities in analysis. The following sharp triangle inequality was given earlier in H. Hudzik and T. R. Landes [2] and also found in a recent paper of L. Maligranda [5].

Theorem 1.1. *For all nonzero elements x, y in a normed linear space X with $\|x\| \geq \|y\|$,*

$$\begin{aligned} \|x+y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|y\| &\leq \|x\| + \|y\| \\ &\leq \|x+y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\|. \end{aligned}$$

We recall that a quasi-norm $\|\cdot\|$ defined on a vector space X (over a real or complex field $\mathbb{K}$) is a map $X \to \mathbb{R}^+$ such that:

(i) $\|x\| > 0$ for $x \neq 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in K, x \in X$;
(iii) $\|x+y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$, where C is a constant independent of x, y.

If $\|\cdot\|$ is a quasi-norm on X defining a complete metrizable topology, then X is called a quasi-Banach space.

In the present paper we will present the triangle inequality in quasi-normed spaces.

157-07

2. MAIN RESULTS

Theorem 2.1. *For all nonzero elements x, y in a quasi-Banach space X with $\|x\| \geq \|y\|$*

$$\|x+y\| + C\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\|$$

$$\leq C(\|x\|+\|y\|) \tag{2.1}$$

$$\leq \|x+y\| + \left(2C^2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|, \tag{2.2}$$

where $C \geq 1$.

Proof. Let $\|x\| \geq \|y\|$. We first show the inequality (2.1).

$$\begin{aligned}
\|x+y\| &= \left\| \|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right) + \|x\|\frac{x}{\|x\|} - \|y\|\frac{x}{\|x\|} \right\| \\
&\leq C\left\| \|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right)\right\| + C\left\|\|x\|\frac{x}{\|x\|} - \|y\|\frac{x}{\|x\|}\right\| \\
&= C\|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| + C(\|x\| - \|y\|) \\
&= C\|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| + C(\|x\| + \|y\| - 2\|y\|) \\
&= C\|y\|\left(\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - 2\right) + C(\|x\| + \|y\|).
\end{aligned}$$

Since

$$\begin{aligned}
\|x+y\| &= \left\| \|x\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right) - \left(\|x\|\frac{y}{\|y\|} - \|y\|\frac{y}{\|y\|}\right) \right\| \\
&\geq \frac{1}{C}\left\| \|x\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right)\right\| - \left\|\|x\|\frac{x}{\|x\|} - \|y\|\frac{x}{\|x\|}\right\| \\
&= \frac{1}{C}\|x\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - (\|x\| - \|y\|) \\
&= \frac{1}{C}\|x\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| + (\|x\| + \|y\| - 2\|x\|) \\
&= \|x\|\left(\frac{1}{C}\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - 2\right) + (\|x\| + \|y\|).
\end{aligned}$$

we have

$$\begin{aligned}
C(\|x\|+\|y\|) &\leq C\|x+y\| + \left(2C - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\| \\
&= \|x+y\| + (C-1)\|x+y\| + \left(2C - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\| \\
&\leq \|x+y\| + (C-1)C(\|x\|+\|y\|) + \left(2C - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|
\end{aligned}$$

$$\leq \|x+y\| + (C-1)C(2\|x\|) + \left(2C - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|$$
$$= \|x+y\| + \left(2C^2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|.$$

Thus the inequality (2.2) holds. □

T. Aoki [1] and S. Rolewicz [6] characterized quasi-Banach spaces as follows:

Theorem 2.2 (Aoki-Rolewicz Theorem)**.** *Let X be a quasi-Banach space. Then there exists $0 < p \leq 1$ and an equivalent quasi-norm $|||\cdot|||$ on X that satisfies for every $x, y \in X$*

$$|||x+y|||^p \leq |||x|||^p + |||y|||^p.$$

Idea of the proof. Let $\|\cdot\|$ be the original quasi-norm on X, denote by $k = \inf\{K \geq 1 :$ for any $x, y \in X, \|x+y\| \leq K(\|x\| + \|y\|)\}$ and p is such that $2^{1/p} = 2k$. It is shown [3] that the function $|||\cdot|||$ defined on X by:

$$|||x||| = \inf\left\{\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i\right\}$$

is an equivalent quasi-norm on X that satisfies the required inequality. □

Next, we will prove the p-triangle inequality in quasi-Banach spaces.

Theorem 2.3. *For all nonzero elements x, y in a quasi-Banach space X with $\|x\| \geq \|y\|$,*

$$\|x+y\|^p + \left(\|x\|^p + \|y\|^p - (\|x\| - \|y\|)^p - \|y\|^p \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^p\right)$$
$$\leq \|x\|^p + \|y\|^p$$
$$\leq \|x+y\|^p + \left(\|x\|^p + \|y\|^p + (\|x\| - \|y\|)^p - \|x\|^p \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^p\right),$$

where $0 < p \leq 1$.

Proof. We have

$$\|x+y\|^p = \left\|\|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right) + \|x\|\frac{x}{\|x\|} - \|y\|\frac{x}{\|x\|}\right\|^p$$
$$\leq \left\|\|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right)\right\|^p + \left\|\|x\|\frac{x}{\|x\|} - \|y\|\frac{x}{\|x\|}\right\|^p$$
$$= \|y\|^p \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^p + (\|x\| - \|y\|)^p$$
$$= \|y\|^p \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^p + \|x\|^p$$
$$+ \|y\|^p - (\|x\|^p + \|y\|^p) + (\|x\| - \|y\|)^p.$$

Thus

$$\|x+y\|^p + \left(\|x\|^p + \|y\|^p - (\|x\| - \|y\|)^p - \|y\|^p \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^p\right) \leq \|x\|^p + \|y\|^p$$

and

$$\begin{aligned}
\|x+y\|^p &= \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) - \left(\|x\| \frac{y}{\|y\|} - \|y\| \frac{y}{\|y\|} \right) \right\|^p \\
&\geq \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\|^p - \left\| \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\|^p \\
&= \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p - (\|x\| - \|y\|)^p \\
&= \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \\
&\qquad + \|x\|^p + \|y\|^p - (\|x\|^p + \|y\|^p) - (\|x\| - \|y\|)^p.
\end{aligned}$$

Hence

$$\|x\|^p + \|y\|^p \leq \|x+y\|^p + \left(\|x\|^p + \|y\|^p + (\|x\| - \|y\|)^p - \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \right).$$

This completes the proof. □

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