

# ON THE TRIANGLE INEQUALITY IN QUASI-BANACH SPACES

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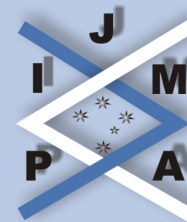
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*Abstract:* In this paper, we show the triangle inequality and its reverse inequality in quasi-Banach spaces.



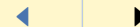
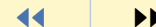
Triangle Inequality in  
Quasi-Banach Spaces

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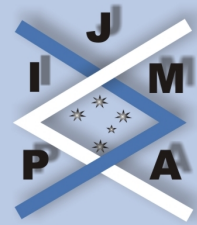
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## 1. Introduction

The triangle inequality is one of the most fundamental inequalities in analysis. The following sharp triangle inequality was given earlier in H. Hudzik and T. R. Landes [2] and also found in a recent paper of L. Maligranda [5].

**Theorem 1.1.** *For all nonzero elements  $x, y$  in a normed linear space  $X$  with  $\|x\| \geq \|y\|$ ,*

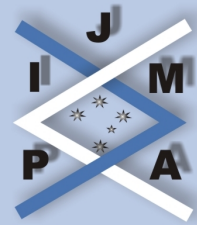
$$\begin{aligned} \|x + y\| + \left( 2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|y\| \\ \leq \|x\| + \|y\| \\ \leq \|x + y\| + \left( 2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\|. \end{aligned}$$

We recall that a quasi-norm  $\|\cdot\|$  defined on a vector space  $X$  (over a real or complex field  $\mathbb{K}$ ) is a map  $X \rightarrow \mathbb{R}^+$  such that:

- (i)  $\|x\| > 0$  for  $x \neq 0$ ;
- (ii)  $\|\alpha x\| = |\alpha| \|x\|$  for  $\alpha \in K, x \in X$ ;
- (iii)  $\|x + y\| \leq C(\|x\| + \|y\|)$  for all  $x, y \in X$ , where  $C$  is a constant independent of  $x, y$ .

If  $\|\cdot\|$  is a quasi-norm on  $X$  defining a complete metrizable topology, then  $X$  is called a quasi-Banach space.

In the present paper we will present the triangle inequality in quasi-normed spaces.



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## 2. Main Results

**Theorem 2.1.** For all nonzero elements  $x, y$  in a quasi-Banach space  $X$  with  $\|x\| \geq \|y\|$

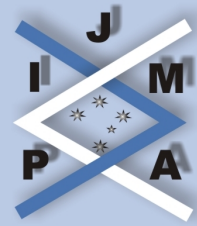
$$(2.1) \quad \begin{aligned} & \|x + y\| + C \left( 2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|y\| \\ & \leq C(\|x\| + \|y\|) \end{aligned}$$

$$(2.2) \quad \leq \|x + y\| + \left( 2C^2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\|,$$

where  $C \geq 1$ .

*Proof.* Let  $\|x\| \geq \|y\|$ . We first show the inequality (2.1).

$$\begin{aligned} \|x + y\| &= \left\| \|y\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) + \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\| \\ &\leq C \left\| \|y\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| + C \left\| \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\| \\ &= C\|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + C(\|x\| - \|y\|) \\ &= C\|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + C(\|x\| + \|y\| - 2\|y\|) \\ &= C\|y\| \left( \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| - 2 \right) + C(\|x\| + \|y\|). \end{aligned}$$



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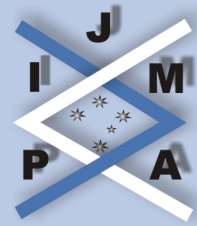
Since

$$\begin{aligned}\|x + y\| &= \left\| \|x\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) - \left( \|x\| \frac{y}{\|y\|} - \|y\| \frac{y}{\|y\|} \right) \right\| \\ &\geq \frac{1}{C} \left\| \|x\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| - \left\| \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\| \\ &= \frac{1}{C} \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| - (\|x\| - \|y\|) \\ &= \frac{1}{C} \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + (\|x\| + \|y\| - 2\|x\|) \\ &= \|x\| \left( \frac{1}{C} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| - 2 \right) + (\|x\| + \|y\|).\end{aligned}$$

we have

$$\begin{aligned}C(\|x\| + \|y\|) &\leq C\|x + y\| + \left( 2C - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| \\ &= \|x + y\| + (C - 1)\|x + y\| + \left( 2C - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| \\ &\leq \|x + y\| + (C - 1)C(\|x\| + \|y\|) + \left( 2C - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| \\ &\leq \|x + y\| + (C - 1)C(2\|x\|) + \left( 2C - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| \\ &= \|x + y\| + \left( 2C^2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\|.\end{aligned}$$

Thus the inequality (2.2) holds. □



T. Aoki [1] and S. Rolewicz [6] characterized quasi-Banach spaces as follows:

**Theorem 2.2 (Aoki-Rolewicz Theorem).** *Let  $X$  be a quasi-Banach space. Then there exists  $0 < p \leq 1$  and an equivalent quasi-norm  $\|\cdot\|$  on  $X$  that satisfies for every  $x, y \in X$*

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p.$$

*Idea of the proof.* Let  $\|\cdot\|$  be the original quasi-norm on  $X$ , denote by  $k = \inf\{K \geq 1 : \text{for any } x, y \in X, \|x + y\| \leq K(\|x\| + \|y\|)\}$  and  $p$  is such that  $2^{1/p} = 2k$ . It is shown [3] that the function  $\|\cdot\|$  defined on  $X$  by:

$$\|x\| = \inf \left\{ \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} : x = \sum_{i=1}^n x_i \right\}$$

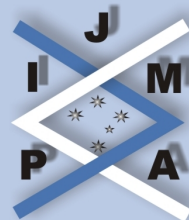
is an equivalent quasi-norm on  $X$  that satisfies the required inequality. □

Next, we will prove the  $p$ -triangle inequality in quasi-Banach spaces.

**Theorem 2.3.** *For all nonzero elements  $x, y$  in a quasi-Banach space  $X$  with  $\|x\| \geq \|y\|$ ,*

$$\begin{aligned} & \|x + y\|^p + \left( \|x\|^p + \|y\|^p - (\|x\| - \|y\|)^p - \|y\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \right) \\ & \leq \|x\|^p + \|y\|^p \\ & \leq \|x + y\|^p + \left( \|x\|^p + \|y\|^p + (\|x\| - \|y\|)^p - \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \right), \end{aligned}$$

where  $0 < p \leq 1$ .



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*Proof.* We have

$$\begin{aligned}
 \|x + y\|^p &= \left\| \|y\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) + \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\|^p \\
 &\leq \left\| \|y\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\|^p + \left\| \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\|^p \\
 &= \|y\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p + (\|x\| - \|y\|)^p \\
 &= \|y\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p + \|x\|^p \\
 &\quad + \|y\|^p - (\|x\|^p + \|y\|^p) + (\|x\| - \|y\|)^p.
 \end{aligned}$$

Thus

$$\|x + y\|^p + \left( \|x\|^p + \|y\|^p - (\|x\| - \|y\|)^p - \|y\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \right) \leq \|x\|^p + \|y\|^p$$

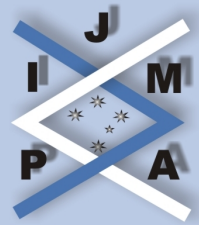
and

$$\begin{aligned}
 \|x + y\|^p &= \left\| \|x\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) - \left( \|x\| \frac{y}{\|y\|} - \|y\| \frac{y}{\|y\|} \right) \right\|^p \\
 &\geq \left\| \|x\| \left( \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\|^p - \left\| \|x\| \frac{x}{\|x\|} - \|y\| \frac{x}{\|x\|} \right\|^p \\
 &= \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p - (\|x\| - \|y\|)^p \\
 &= \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \\
 &\quad + \|x\|^p + \|y\|^p - (\|x\|^p + \|y\|^p) - (\|x\| - \|y\|)^p.
 \end{aligned}$$

Hence

$$\|x\|^p + \|y\|^p \leq \|x+y\|^p + \left( \|x\|^p + \|y\|^p + (\|x\| - \|y\|)^p - \|x\|^p \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^p \right).$$

This completes the proof.  $\square$



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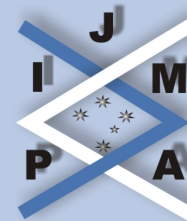
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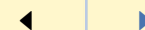
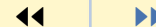
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