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THE BOUNDARY OF WEIGHTED ANALYTIC CENTERS FOR LINEAR MATRIX INEQUALITIES

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ABSTRACT. We study the boundary of the region of weighted analytic centers for linear matrix inequality constraints. Let be the convex subset of $\mathbb{R}^n$ defined by q simultaneous linear matrix inequalities (LMIs)

$$A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succ 0, \;\; j = 1, 2, \dots, q,$$

where $A_i^{(j)}$ are symmetric matrices and $x \in \mathbb{R}^n$. Given a strictly positive vector $\omega = (\omega_1, \omega_2, \dots, \omega_q)$, the *weighted analytic center* $x_{ac}(\omega)$ is the minimizer of the strictly convex function

$$\phi_\omega(x) := \sum_{j=1}^{q} \omega_j \log\det[A^{(j)}(x)]^{-1}$$

over $\mathcal{R}$. The region of weighted analytic centers, $\mathcal{W}$, is a subset of $\mathcal{R}$. We give several examples for which $\mathcal{W}$ has interesting topological properties. We show that every point on a central path in semidefinite programming is a weighted analytic center.

We introduce the concept of the *frame* of $\mathcal{W}$, which contains the boundary points of $\mathcal{W}$ which are not boundary points of $\mathcal{R}$. The frame has the same dimension as the boundary of $\mathcal{W}$ and is therefore easier to compute than $\mathcal{W}$ itself. Furthermore, we develop a Newton-based algorithm that uses a Monte Carlo technique to compute the frame points of $\mathcal{W}$ as well as the boundary points of $\mathcal{W}$ that are also boundary points of $\mathcal{R}$.

Key words and phrases: Linear matrix inequalities, Analytic center, Central path, Semidefinite programming.

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1. INTRODUCTION

The study of Linear Matrix Inequalities (LMIs) is important in semidefinite programming ([25], [1], [27], [24]). A semidefinite programming problem (SDP) contains an objective function to be optimized subject to a system of linear matrix inequality (LMI) constraints. SDPs arise among others in relaxations of combinatorial optimization problems, in control theory, in solving structural design problems and in statistics.

The interest in weighted analytic centers arises from their success in solving linear programming problems ([19], [21]). The study of weighted analytic center continues to be of interest in semidefinite programming because of its connection to the central path (see [4], [23], [15], [12] and [16]). Most interior point methods in semidefinite programming follow the central path. A recent paper [17] gives an extension of weighted analytic center for linear programming ([3], [12], [18]) to semidefinite constraints, and shows that the region of weighted analytic centers is not convex in $\mathbb{R}^3$. Our paper can be considered as an extension of [17].

For a symmetric, real matrix A, define $A \succ 0$ to mean that A is positive definite, and $A \succeq 0$ to mean that A is positive semidefinite. Consider the following system of q Linear Matrix Inequality (LMI) constraints:

$$A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succ 0, \;\; j = 1, 2, \dots, q, \tag{1.1}$$

where $A_i^{(j)},\ 0 \le i \le n$, are square symmetric matrices of size m_j. Let

$$\mathcal{R} = \left\{ x \in \mathbb{R}^n : A^{(j)}(x) \succ 0,\ j = 1, 2, \dots, q \right\}$$

be the *feasible region* of the LMI constraints. Note that $\det A > 0$ is a necessary, but not sufficient, condition for $A \succ 0$. Therefore, $\det A^{(j)}(x) > 0$ for all $x \in \mathcal{R}$ and all $j \in \{1, 2, \dots, q\}$. Furthermore, if $x \in \partial\mathcal{R}$, the boundary of $\mathcal{R}$, then $\det A^{(j)}(x) = 0$ for some $j \in \{1, 2, \dots, q\}$. It is well known that $\mathcal{R}$ is convex [25].

Note that $\mathcal{R}$ is open, since we require that the $A^{(j)}(x)$ are positive definite. In SDP, $\mathcal{R}$ is often the closed set where the matrices are positive semidefinite (see [25]). However, interior point methods use the interior of the usual $\mathcal{R}$, which is our set $\mathcal{R}$.

We shall show that the following assumption is necessary and sufficient for the existence and uniqueness of the weighted analytic center.

Assumption 1. The feasible region $\mathcal{R}$ is nonempty and bounded.

Our Assumption 1 differs from Assumption 1.1 of [17]. In particular, we do not assume that $q > n$, and we do not assume that there are n linearly independent gradients of the constraints at every $x \in \mathcal{R}$. One of the main objectives of the current paper is to explore the consequences of dropping these two assumptions. The latter assumption, about linear independence, is difficult to check and somewhat unnatural.

Let $\mathbb{R}_+ := (0, \infty)$ be the set of *positive* real numbers. Given a weight vector $\omega \in \mathbb{R}_+^q$, the *weighted analytic center* $x_{ac}(\omega)$ determined by the q LMI constraints is given by the unique optimal solution of the *maxdet* problem [17]

$$\min \phi_\omega(x) := \sum_{j=1}^{q} \omega_j \log\det A^{(j)}(x)^{-1} \;\text{ s.t. } A^{(j)}(x) \succ 0, \;\; j = 1, 2, \dots, q. \tag{1.2}$$

The weighted analytic center exists, and is unique, given our Assumption 1. This is shown in [17] under stronger assumptions, which we will show are implied by our Assumption 1.

The function $\phi_\omega(x)$ is a *barrier function*, or a *potential function*. It is a barrier function because it gets infinitely large if one starts from the interior and goes towards the boundary of $\mathcal{R}$.

The set of all points in $\mathcal{R}$ that are weighted analytic centers for some ω is called the *region of weighted analytic centers* $\mathcal{W}$. That is,

$$\mathcal{W} = \{x_{ac}(\omega) \; : \; \omega \in \mathbb{R}^q_+\} \subseteq \mathcal{R}.$$

Unlike the special case of linear inequalities, $\mathcal{W}$ does not equal $\mathcal{R}$ in general.

It was shown in [17] that $x_{ac}(\omega)$ is analytic, using the implicit function theorem. Therefore $\mathcal{W}$ is the image of the open set $\mathbb{R}^q_+$ under an analytic map. It follows that $\mathcal{W}$ is connected, but we cannot say much more. We show by an example that $\mathcal{W}$ is not in general convex in $\mathbb{R}^2$. This implies that $\mathcal{W}$ is not necessarily convex in $\mathbb{R}^n$ for $n \geq 2$. We show that $\mathcal{W}$ is not open if $q \leq n$. We also show by an example that $\mathcal{W}$ is not necessarily open if $q > n$.

We show that the region $\mathcal{W}$ of weighted analytic centers of [17] extends the central path used by most SDP solvers such as SeDuMi [24]. Our definition of the weighted analytic center has the added advantage that it can be used to define the concept of repelling path and repelling limits of LMI constraints. A different approach to the notion of weighted analytic center is given in [23] and [15]. The concept of repelling paths in linear programming was first introduced in [5]. They showed that a repelling path, as a function of the barrier parameter μ, has a unique limit as $\mu \to \infty$. We extend the notion of repelling paths and repelling limits from linear programming to semidefinite programming.

The limiting behavior of the central path in semidefinite programming has been studied recently in ([7], [13], [9]). Under an assumption of strict feasibility, Goldfarb and Scheinberg [7] show that the central path exists and converges to the analytic center of the optimal solution set of the SDP. The first correct proof (assuming strict complementarity) is due to Luo et al [13]. Halicka re-derived this result in [9]. We show that repelling paths in semidefinite programming are analytic and the repelling limits are not necessarily on the boundary of the feasible region.

The main result of this paper is the *WF Algorithm*, which finds the boundary points of $\mathcal{W}$ by a Monte Carlo method. This approach is more efficient than finding the boundary of $\mathcal{W}$ by computing points in $\mathcal{W}$ randomly, since we do not compute most of the interior points. The WF Algorithm approximates repelling limits to compute the boundary of $\mathcal{W}$, along with a few more points, which together we call *frame points*. Our concept of the frame of $\mathcal{W}$, which contains the boundary points of $\mathcal{W}$ that are in $\mathcal{R}$, is the main theoretical contribution of this paper.

The WF algorithm also finds boundary points of $\mathcal{W}$ which are not in $\mathcal{R}$. It uses a modification of Newton's method that aids convergence to points extremely close to the boundary of $\mathcal{R}$: If Newton's method sends a point out of $\mathcal{R}$, we move half-way to the boundary of $\mathcal{R}$ from the current iterate along the search direction.

2. BACKGROUND

In this section, we present some basic results on the optimization problem (1.2), most of which are found in [17]. For a fixed weight vector $\omega \in \mathbb{R}^q_+$, the weighted analytic center is the unique point in $\mathcal{R}$ at which the gradient of $\phi_\omega(x)$ is 0. Thus, the weighted analytic center $x_{ac}(\omega)$ is the solution to the n equations in the n unknowns $x = (x_1, x_2, \ldots, x_n)$:

$$\nabla_i \phi_\omega(x) = -\sum_{j=1}^{q} \omega_j \frac{\nabla_i |A^{(j)}(x)|}{|A^{(j)}(x)|} = 0 \text{ for } i = 1, 2, \ldots, n, \tag{2.1}$$

where $\nabla_i = \frac{\partial}{\partial x_i}$. To derive (2.1), we used the fact that $\log|A^{-1}| = \log(1/|A|) = -\log|A|$. Note that the factor multiplying ω_j in equation (2.1) is a rational function in x, since $|A^{(j)}(x)|$ and its partial derivatives are polynomials in x.

There is a unique solution (in $\mathcal{R}$) to the system of equations (2.1), since $\phi_\omega(x)$ is strictly convex (see [17], and Lemma 2.3 below), and the potential $\phi_\omega(x)$ grows without bound as x approaches the boundary of $\mathcal{R}$. (Note that the system (2.1) can have other solutions outside $\mathcal{R}$.) The equations (2.1) are defined provided $|A^{(j)}| \neq 0$ for all j.

The Hessian matrix of $\phi_\omega(x)$ is the Jacobian matrix of the system of equations (2.1). This Jacobian matrix is used in Newton's method, and it is also important in the implicit function theorem. A formula to compute the gradient and the Hessian of $\phi_\omega(x)$, without symbolic differentiation, is given in [17]. See also [4].

Lemma 2.1. *For $\phi_\omega(x)$ defined in (1.2) and $x \in \mathcal{R}$*

$$\nabla_i \phi_\omega(x) = -\sum_{j=1}^{q} \omega_j (A^{(j)}(x))^{-1} \bullet A_i^{(j)}$$

$$H_\omega(x)_{ij} := \nabla^2_{ij}\phi_\omega(x) = \sum_{k=1}^{q} \omega_k [(A^{(k)}(x))^{-1} A_i^{(k)}]^T \bullet [(A^{(k)}(x))^{-1} A_j^{(k)}],$$

where $\nabla^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and the inner product, $\bullet$, on square matrices is defined as

$$A \bullet B = Tr(A^T B) = \sum_{i,\, j=1}^{m} A_{ij} B_{ij}.$$

Note that there is a typographic error in the formula for $H_\omega(x)$ in [17]; the transpose is missing. The expression for the Hessian can be written in a way that involves the inner product of *symmetric* matrices,

$$H_\omega(x)_{ij} = \sum_{k=1}^{q} \omega_k [(A^{(k)}(x))^{-1/2} A_i^{(k)} (A^{(k)}(x))^{-1/2}] \bullet [(A^{(k)}(x))^{-1/2} A_j^{(k)} (A^{(k)}(x))^{-1/2}]. \tag{2.2}$$

Assumption 2.1 of [17] is that the matrices $\{A_{<1>}, A_{<2>}, \ldots, A_{<q>}\}$ are linearly independent. This assumption can be replaced by the assumption that $\mathcal{R}$ is bounded and nonempty due to the following lemma.

Lemma 2.2. *If the feasible region $\mathcal{R}$ of (1.1) is bounded and nonempty, then the matrices*

$$A_{<i>} = \mathbf{diag}[A_i^{(1)}, A_i^{(2)}, \cdots, A_i^{(q)}] \text{ for } i = 0, 1, 2, \ldots, n$$

are linearly independent. Equivalently, if $\mathcal{R}$ is bounded and nonempty there is no nonzero $s \in \mathbb{R}^n$ such that $\sum_{i=1}^n s_i A_i^{(j)} = 0$ for all $j \in \{1, 2, \cdots, q\}$.

Proof. Assume $x^* \in \mathcal{R}$, and assume that the $A_{<i>}$ are linearly dependent. We must show that $\mathcal{R}$ is unbounded. Since the $A_{<i>}$ are linearly dependent, there is a nonzero vector $s \in \mathbb{R}^n$ such that $\sum_{i=1}^n s_i A_{<i>} = 0$. Therefore $\sum_{i=1}^n s_i A_i^{(j)} = 0$ for each $j \in \{1, 2, \ldots, q\}$. Thus,

$$A^{(j)}(x^* + \sigma s) = A_0^{(j)} + \sum_{i=1}^{n} A_i^{(j)}(x^* + \sigma s) = A^{(j)}(x^*) + \sigma \sum_{i=1}^{n} s_i A_i^{(j)} = A^{(j)}(x^*)$$

for all real σ. But $A^{(j)}(x^*) \succ 0$ since $x^* \in \mathcal{R}$. Therefore, $x^* + \sigma s \in \mathcal{R}$ for all $\sigma \in \mathbb{R}$, and $\mathcal{R}$ is unbounded. $\square$

The following lemma is well known for the case $q = 1$ (see [4]). The extension to $q > 1$ was proved in [17].

Lemma 2.3. [17] *Assume that $\mathcal{R}$ is nonempty and bounded. Then the Hessian matrix $H_\omega(x)$ is positive definite for all $x \in \mathcal{R}$ and all $\omega \in \mathbb{R}^q_+$. Hence, $\phi_\omega(x)$ is strictly convex over $\mathcal{R}$.*

Remark 2.4. Lemma 2.3 is false if we remove the hypothesis that $\mathcal{R}$ is bounded. For example, consider the 2 linear inequalities in $\mathbb{R}^2$: $x_1 > 0$ and $1 - x_1 > 0$. Then $\phi_\omega(x_1, x_2) = -\omega_1 \log(x_1) - \omega_2 \log(1 - x_1)$ is not *strictly* convex in $\mathcal{R} = \{(x_1, x_2)|\ 0 < x_1 < 1\}$, which is an unbounded strip in $\mathbb{R}^2$. Furthermore, Lemma 2.3 is false if $\omega \geq 0$ replaces $\omega > 0$ ($\omega \in \mathbb{R}^q_+$). Consider the 3 linear inequalities in $\mathbb{R}^2 : x_1 > 0$, $1 - x_1 > 0$ and $x_1^2 + x_2^2 < 1$. If $\omega_3 = 0$, then ϕ_ω is not *strictly* convex in $\mathcal{R}$.

It is evident from the structure of (1.2) and (2.1) that the weighted analytic center is the same for the weight vector ω and any positive scalar multiple, $k\omega$. That is,

$$x_{ac}(\omega) = x_{ac}(k\omega) \text{ for all } k > 0.$$

Therefore the set of weights can be constrained to the *open simplex*

$$\Delta^{q-1} := \left\{ \omega \in \mathbb{R}^q_+ : \sum_{j=1}^{q} \omega_j = 1 \right\}.$$

Note that Δ^{q-1} is open in the $(q-1)$-dimensional affine subspace of $\mathbb{R}^q$ defined by $\sum_{j=1}^q \omega_j = 1$, but it is not open as a subset of $\mathbb{R}^q$. The region of weighted analytic centers can be described in two ways:

$$\mathcal{W} = \{x_{ac}(\omega) : \omega \in \mathbb{R}^q_+\} = \{x_{ac}(\omega) : \omega \in \Delta^{q-1}\}.$$

The following lemma describes how we choose random weight vectors in our numerical experiments.

Lemma 2.5. *Let $\omega_j = -\log(u_j)$ independently for each $j \in \{1, 2, \dots, q\}$, where u_j is chosen from a uniform distribution on $(0, 1)$. Then, the normalized weight vectors*

$$\widetilde{\omega} = \frac{\omega}{\sum_{j=1}^q \omega_j}$$

are uniformly distributed on the open simplex Δ^{q-1}.

Proof. The probability density function for each ω_j is $f(\omega_j) = e^{-\omega_j}$, where $\omega_j > 0$. Hence the probability density function on ω is

$$f(\omega) = e^{-\omega_1} e^{-\omega_2} \cdots e^{-\omega_q} = e^{-\sum \omega_j}.$$

Therefore, the probability density of $\omega \in \mathbb{R}^q_+$ is constant on each slice where $\sum_{j=1}^q \omega_j$ is constant. It follows that the probability density of $\widetilde{\omega}$ in Δ^{q-1} is constant. □

The reason $\phi_\omega(x)$ is called a potential function comes from physics. We define

$$\phi^{(j)}(x) := \log\det A^{(j)}(x)^{-1} = -\log|A^{(j)}(x)|$$

to be the *potential energy* associated with the j^{th} constraint. The negative gradient of this potential energy is a *boundary force* $\mathbf{F}^{(j)}(x)$ pushing away from the boundary:

$$\mathbf{F}^{(j)}(x) := -\nabla\phi^{(j)}(x) = \frac{\nabla|A^{(j)}(x)|}{|A^{(j)}(x)|}.$$

It is clear that the components of this boundary force are rational functions of x, since $|A^{(j)}(x)|$ is a polynomial.

As x approaches the j^{th} boundary, the magnitude of the force $\mathbf{F}^{(j)}(x)$ grows without bound, and the force points normal to boundary, into $\mathcal{R}$.

The system of equations for the minimizer of $\phi_\omega(x)$, equation (2.1), says that the weighted vector sum of the boundary forces, defined to be $\mathbf{F}_\omega(x)$, is the zero vector:

$$\mathbf{F}_\omega(x) := \sum_{j=1}^{q} \omega_j \, \mathbf{F}^{(j)}(x) = 0.$$

Therefore, the region of weighted analytic centers can be characterized as

$$\mathcal{W} = \{x \in \mathcal{R} : \text{ there exists } \omega \in \mathbb{R}^q_+ \text{ such that } \mathbf{F}_\omega(x) = 0\}.$$

In other words, a point x is in $\mathcal{W}$ if and only if the force vectors $\mathbf{F}^{(j)}(x)$ are *positively linearly dependent*. Geometrically, a finite set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ is positively linearly dependent if and only if the zero vector is in their *positive convex hull*, defined to be

$$\operatorname{conv}^+(\mathcal{S}) = \left\{ \sum_{j=1}^{q} \omega_j \mathbf{F}^{(j)} : \omega_j > 0, \sum_{j=1}^{q} \omega_j = 1, \; \mathbf{F}^{(j)} \in \mathcal{S} \right\}.$$

The positive convex hull is precisely the relative interior of the convex hull of $\mathcal{S}$, denoted $\operatorname{conv}(\mathcal{S})$ (see [8]). Let $\mathcal{S}$ be a finite set of points in $\mathbb{R}^n$. Then $\operatorname{conv}^+(\mathcal{S})$ is an open subset of $\mathbb{R}^n$ if and only if the interior of $\operatorname{conv}(\mathcal{S})$ is nonempty. Furthermore, the interior of $\operatorname{conv}(\mathcal{S})$ is nonempty if and only if there is some subset of $n+1$ vectors which are affinely independent [8]. Recall that a set of vectors is *affinely dependent* if and only if there is a set of weights α_j, not all zero, such that

$$\sum_{j=1}^{q} \alpha_j \mathbf{F}^{(j)} = 0 \text{ and } \sum_{j=1}^{q} \alpha_j = 0.$$

Theorem 2.6. *If $x^* \in \mathcal{W}$, and $\{\mathbf{F}^{(j)}(x^*) : \ j = 1, 2, \ldots, q\}$ spans $\mathbb{R}^n$, then x^* is an interior point of $\mathcal{W}$.*

Proof. Assume that $x^* \in \mathcal{W}$, and let $\mathcal{S} = \{\mathbf{F}^{(j)}(x^*) : \ j = 1, 2, \ldots, q\}$. Assume that $\operatorname{span}(\mathcal{S}) = \mathbb{R}^n$. Therefore there is a set $\mathcal{S}' \subseteq \mathcal{S}$ consisting of n linearly independent vectors. Since $x^* \in \mathcal{W}$, we know that $0 \in \operatorname{conv}^+(\mathcal{S})$. Therefore $\operatorname{conv}^+(\mathcal{S}) = \operatorname{conv}^+(\mathcal{S} \cap \{0\})$. Now, the set $\mathcal{S}' \cap \{0\}$ is a set of $n+1$ affinely independent vectors, so $\operatorname{conv}^+(\mathcal{S}' \cap \{0\})$ is open. Therefore, this convex hull has positive n-dimensional volume, and it follows that $\operatorname{conv}^+(\mathcal{S})$ is open. Since the force vectors depend continuously on x, there is a neighborhood U of x^* such that $0 \in \operatorname{conv}^+\{\mathbf{F}^{(j)}(x) : \ j = 1, 2, \ldots, q\}$ for all $x \in U$. Therefore $x \in \mathcal{W}$ for all $x \in U$, and x^* is an interior point of $\mathcal{W}$. □

The following corollary gives conditions which ensure that $\mathcal{W}$ is open. This was proved by a different method in [17]. The statement of the theorem in [17] has the hypothesis that there is a set of n linearly independent force vectors at every point, which is equivalent to our hypothesis that the force vectors span $\mathbb{R}^n$.

Corollary 2.7. *If the set of force vectors $\{\mathbf{F}^{(j)}(x) : \ j = 1, 2, \ldots, q\}$ spans $\mathbb{R}^n$ at every point $x \in \mathcal{R}$, then the set of weighted analytic centers $\mathcal{W}$ is open.*

Proposition 2.8. *If $x^* \in \mathcal{W}$ and $q \leq n$, then the set $\{\mathbf{F}^{(j)}(x^*) : \ j = 1, 2, \ldots, q\}$ does not span $\mathbb{R}^n$.*

Proof. If $q < n$ the q force vectors cannot span $\mathbb{R}^n$. Assume, by way of contradiction, that $q = n$, $x^* \in \mathcal{W}$ and the set of force vectors spans $\mathbb{R}^n$. The force vectors must be linearly independent, so no nontrivial linear combination gives the zero vector. On the other hand, since $x^* \in \mathcal{W}$, a linear combination with positive weights gives the zero vector. This is a contradiction. □

Remark 2.9. By Proposition 2.8, Corollary 2.7 does not apply if $q \leq n$. In fact, $\mathcal{W}$ is not open if $q \leq n$, since $\mathcal{W}$ is the continuous image of the set Δ^{q-1}, which has dimension less than n.

3. EXAMPLES AND PROPERTIES OF REGIONS OF WEIGHTED ANALYTIC CENTERS

We give four examples of systems of Linear Matrix Inequalities (LMIs) and their associated feasible region $\mathcal{R}$ and region of weighted analytic centers $\mathcal{W} \subseteq \mathcal{R}$. The examples illustrate some properties of the region of weighted analytic centers $\mathcal{W}$. They will reappear later.

Example 3.1. Consider the system of three linear inequalities (a special case of linear matrix inequalities, with 1×1 matrices):

$$A^{(1)}(x) := x_1 > 0, \;\; A^{(2)}(x) := x_2 > 0, \;\; A^{(3)}(x) := 1 - x_1 - x_2 > 0. \tag{3.1}$$

In this case the feasible region is a triangle, and the three boundary forces are

$$\mathbf{F}^{(1)}(x) = \frac{\mathbf{e}_1}{x_1}, \;\; \mathbf{F}^{(2)}(x) = \frac{\mathbf{e}_2}{x_2}, \;\; \mathbf{F}^{(3)}(x) = \frac{-\mathbf{e}_1 - \mathbf{e}_2}{1 - x_1 - x_2}.$$

Since the forces $\mathbf{F}^{(j)}(x)$ point in the directions $\mathbf{e}_1$, $\mathbf{e}_2$ and $-\mathbf{e}_1 - \mathbf{e}_2$ at every $x \in \mathcal{R}$, it is possible

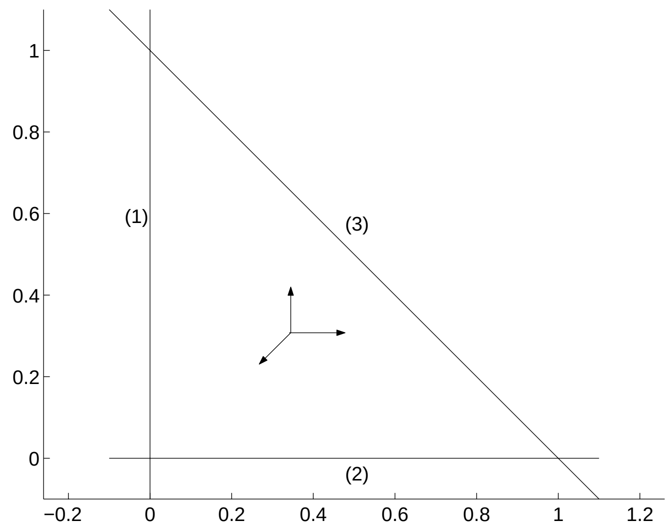

Figure 3.1: The feasible region $\mathcal{R}$ for system (3.1) is a triangle. The direction of the three boundary forces at any point in $\mathcal{R}$ is shown. The region of weighted analytic centers is $\mathcal{W} = \mathcal{R}$.

to choose a positive weight vector to balance the forces. Hence the region of weighted analytic centers is the same as the feasible region: $\mathcal{W} = \mathcal{R}$.

It is true in general that $\mathcal{W} = \mathcal{R}$ for systems of linear inequalities with a bounded feasible region. For systems of linear *matrix* inequalities the situation is more interesting. The next example illustrates the fact that $\mathcal{W}$ is not open if $q \leq n$, since $\mathcal{W}$ is the image of the $q - 1$ dimensional set Δ^{q-1} under the x_{ac} map.

Example 3.2. Consider the system of two linear matrix inequalities in $\mathbb{R}^2$

$$A^{(1)}(x) := 1 + x_1 > 0,$$
$$A^{(2)}(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \succ 0.$$

The feasible region is $x_1^2 + x_2^2 < 1$. The first boundary force always points to the right. The

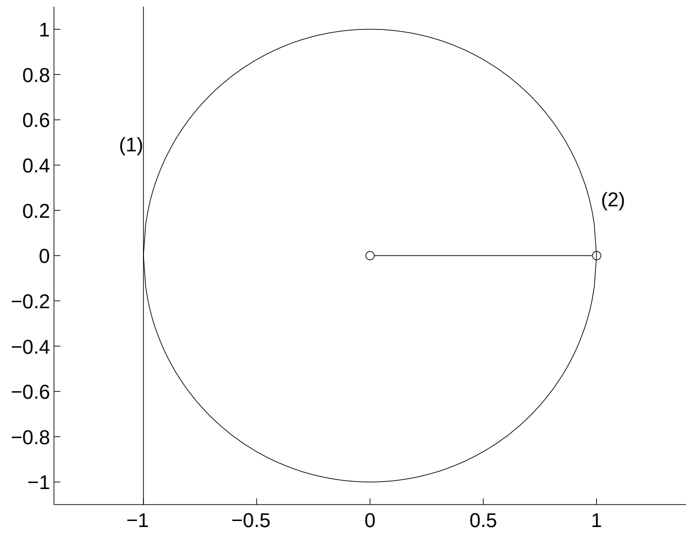

Figure 3.2: The feasible region $\mathcal{R}$ is the open unit disk. The region $\mathcal{W}$ is the line segment without the end points.

second boundary force points toward the center of the circle. Hence, the region of weighted analytic centers is the line segment $\{(x_1, x_2) : 0 < x_1 < 1$ and $x_2 = 0\}$. Note how the addition of redundant constrains (like 1) can influence the region of weighted analytic centers. In this example $\mathcal{W}$ has no interior points. This is always the case when $q \leq n$.

A point x is called a *boundary point* of $A \subseteq \mathbb{R}^n$, if every neighborhood of x contains a point in A and a point not in A. The boundary of A, denoted ∂A, is the set of all boundary points of A. These definitions are standard. An open set like the feasible region $\mathcal{R}$ contains none of its boundary points: $\partial\mathcal{R} \cap \mathcal{R} = \varnothing$. In Example 3.1, the boundary of $\mathcal{R}$ is made of the three line segments of the triangle, and $\partial\mathcal{W} = \partial\mathcal{R}$. In Example 3.2, all points of $\mathcal{W}$ belong to $\partial\mathcal{W}$, but $\partial\mathcal{W}$ also contains $(0, 0)$ and $(1, 0)$, which are not in $\mathcal{W}$.

Example 3.3. For each q, a positive integer, consider the system of linear matrix inequalities:

$$A^{(j)}(x) = \begin{bmatrix} 3 - \cos\left(\frac{2\pi j}{q}\right) & -\sin\left(\frac{2\pi j}{q}\right) \\ -\sin\left(\frac{2\pi j}{q}\right) & 3 + \cos\left(\frac{2\pi j}{q}\right) \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \succ 0 \text{ for } j = 1, \ldots, q.$$

The feasible region $\mathcal{R}$ of Example 3.3 is the intersection of open disks of radius 3 centered at $\left(\cos\left(\frac{2\pi j}{q}\right), \sin\left(\frac{2\pi j}{q}\right)\right)$. The region of weighted analytic centers $\mathcal{W}$ is the positive convex hull of the centers of the disks. Figure 3.3 is the picture for the case $q = 5$. The boundary of the feasible region was found using the *SCD* algorithm described in [11].

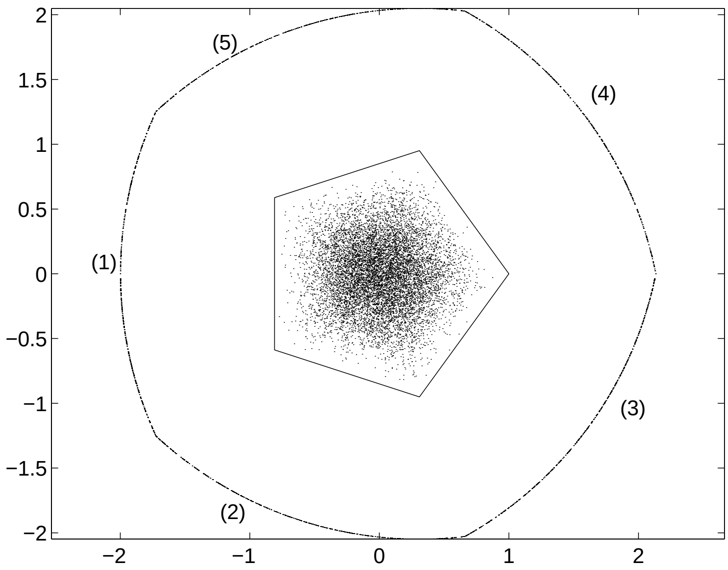

Figure 3.3: The region $\mathcal{W}$ of Example 3.3 for $q = 5$. The dots are the weighted analytic centers for 10,000 weight vectors chosen randomly from the uniform distribution in the simplex Δ^4 described in Lemma 2.5. We used the WF algorithm, listed in the Appendix, with $\mu = 1$. The boundary of $\mathcal{W}$ is the pentagon.

Remark 3.1. Finding the boundary points of $\mathcal{W}$: Figure 3.3 demonstrates that it would require a very large number of random points in $\mathcal{W}$ to get a reasonable number of points near the boundary of $\mathcal{W}$, especially when q is larger than n. In Section 5 we give a method of computing the boundary of $\mathcal{W}$, which focuses on the distinction between feasible and non-feasible boundary points. A boundary point $x^* \in \partial\mathcal{W}$ is *feasible* if $x^* \in \mathcal{R}$ and *infeasible* if $x^* \notin \mathcal{R}$. In Example 3.1, all boundary points of $\mathcal{W}$ are infeasible. In Example 3.2, $(1, 0)$ is an infeasible boundary point while $(x, 0)$ for $0 \le x < 1$ are feasible boundary points. In Example 3.3, all boundary points are feasible.

It is well-known that $\mathcal{R}$ is convex. It is natural to ask if $\mathcal{W}$ is convex. Example 2 of [17] shows that $\mathcal{W}$ is not necessarily convex. In that example, each of the $q = 4$ constraints involves a 5×5 matrix in $n = 3$ variables. A simpler example follows.

Example 3.4. Consider the feasible region $\mathcal{R}$ and the region of weighted analytic centers $\mathcal{W}$ for these $q = 3$ LMI constraints in $n = 2$ variables:

$$A^{(1)}(x) = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \succ 0$$

$$A^{(2)}(x) = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \succ 0$$

$$A^{(3)}(x) = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succ 0.$$

Figure 3.4 shows clearly that $\mathcal{W}$ is not convex. The boundary points $e = (0, 1)$ and $f = (1, 0)$ are the centers of the ellipses where $|A^{(1)}(x)| = 0$ and $|A^{(2)}(x)| = 0$, respectively. Furthermore, $\mathbf{F}^{(1)}(e) = 0$ and $\mathbf{F}^{(2)}(f) = 0$. The boundary points between e and f are where the forces $\mathbf{F}^{(1)}(x)$ and $\mathbf{F}^{(2)}(x)$ are positively linearly dependent, which can be computed to be $x_2 = \frac{1-x_1}{1-15x_1/16}$. The forces $\mathbf{F}^{(1)}(x)$ and $\mathbf{F}^{(3)}(x)$ are positively linearly dependent between e and c, which is a line of slope 1/4. The points between a and f are where the forces $F^{(2)}(x)$ and $F^{(3)}(x)$ are positively

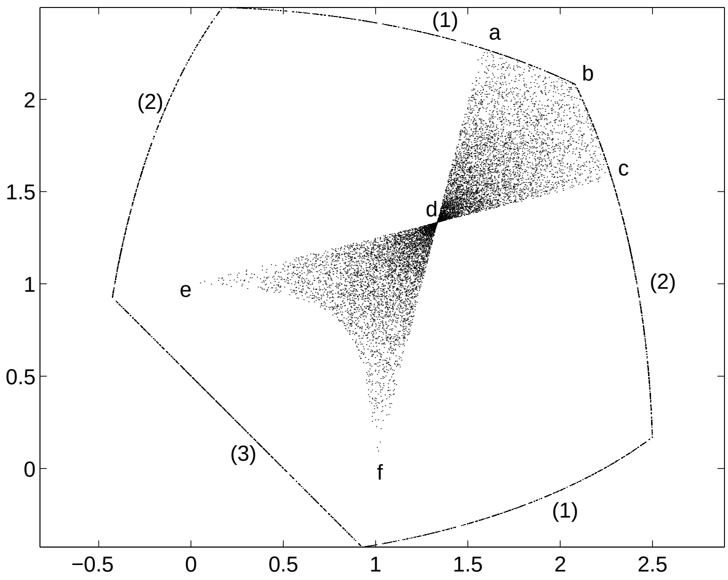

Figure 3.4: The region $\mathcal{W}$ of Example 3.4 given by 10,000 random points as in Figure 3.3. The corner point e is the center of ellipse 1; similarly f is the center of ellipse 2.

linearly dependent, which is a line of slope 4. The point $d = (4/3, 4/3)$ is not an interior point of $\mathcal{W}$, because $d \in \mathcal{W}$ and $d \in \partial\mathcal{W}$. Hence, $\mathcal{W}$ is not open in this example. The fact that d is not an interior point shows that the hypothesis that the force vectors span $\mathbb{R}^n$ is needed in Theorems 2.6 and 2.7. At point d, all of the force vectors are scalar multiples of $\mathbf{e}_1 + \mathbf{e}_2$, so they do not span $\mathbb{R}^2$.

4. REPELLING PATHS AND REPELLING LIMITS

In this section we study the mapping of the open simplex Δ^{q-1} to the region of weighted analytic centers. We also study repelling paths and limits. This extends the concept of repelling limits given in [5].

Define the function $f : \mathcal{R} \times \mathbb{R}^q_+ \mapsto \mathbb{R}^n$, where

$$f_i(x, \omega) = \sum_{k=1}^{q} \omega_k \mathbf{F}_i^{(k)}(x).$$

The following lemma is a reinstatement of Theorems 3.6 and 3.7 of [17]. The proof uses the Implicit Function Theorem in [20].

Lemma 4.1. *The map $x_{ac} : \mathbb{R}^q_+ \to \mathcal{R}; \omega \mapsto x_{ac}(\omega)$ is analytic. Furthermore, the partial derivatives of the weighted analytic center function evaluated at $x = x_{ac}(\omega)$ are:*

$$\frac{\partial x_{ac}(\omega)_i}{\partial \omega_k} = -|H_\omega(x)|^{-1} \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, x_{i-1}, \omega_k, x_{i+1}, \ldots, x_n)}, \tag{4.1}$$

and satisfy:

$$\sum_{j=1}^{n} H_\omega(x)_{ij} \frac{\partial x_{ac}(x)_j}{\partial \omega_k} = -\mathbf{F}_i^{(k)}(x). \tag{4.2}$$

Remark 4.2. If one wishes to compute the partial derivatives of x_{ac}, the system (4.2) should be solved, rather than using equation (4.1).

Corollary 4.3. *The restricted mapping* $x_{ac}|_{\Delta^{q-1}} : \Delta^{q-1} \longrightarrow \mathcal{R}$ *is analytic in* ω.

Proof. This follows from Lemma 4.1, since Δ^{q-1} is an affine subspace of $\mathbb{R}^q_+$. □

The next example shows that we cannot extend $x_{ac}|_{\Delta^{q-1}}$ to an analytic, or even continuous function mapping the closure of the simplex to the closure of $\mathcal{R}$.

Example 4.1. Consider the system with 4 constraints in $\mathbb{R}^2$:

$$A^{(1)} = x_1 > 0,\ A^{(2)} = 1 - x_1 > 0,\ A^{(3)} = x_2 > 0,\ A^{(4)} = 1 - x_2 > 0.$$

The weighted analytic center is at

$$x_1 = \frac{\omega_1}{\omega_1 + \omega_2},\ x_2 = \frac{\omega_3}{\omega_3 + \omega_4}.$$

Note that

$$\lim_{\omega_1 \to \infty} x_{ac}(\omega) = \left(1, \frac{\omega_3}{\omega_3 + \omega_4}\right),$$

which depends on ω_3 and ω_4. But in the simplex the normalized weights approach $(1, 0, 0, 0)$ as $\omega_1 \to \infty$ with the other weights fixed. Therefore $x_{ac}(\omega)$ has no limit as $\omega \in \Delta^{q-1}$ approaches $(1, 0, 0, 0)$. Therefore, $x_{ac} : \Delta^{q-1} \to \mathcal{R}$ cannot be extended to a continuous function at $(1, 0, 0, 0) \in \overline{\Delta^{q-1}}$.

We now show how weighted analytic centers generalize the central path in semidefinite programming. Most interior point algorithms for solving SDP approach the optimal solution by following a central path. Consider the semidefinite programming problem (SDP)

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succ 0, \quad j = 1, 2, \ldots, q. \end{aligned}$$

Let $\mathcal{R}$ denote the feasible region. The central path associated with the SDP is defined by

$$x(\mu) = \operatorname{argmin} \left\{ \mu c^T x + \sum_{j=1}^{q} \log\det\ A^{(j)}(x)^{-1} : x \in \mathcal{R} \right\}.$$

As $\mu \to \infty$, more weight is put on the objective term compared to the barrier term, and $x(\mu)$ approaches the optimal solution of the SDP if strict complementarity holds [13].

We can replace the linear objective function with a redundant constraint as follows. Choose K such that $c^T x < K$ for all x in the (bounded) feasible region $\mathcal{R}$. Then, following the method of Renegar [19] for linear programming, as in [5], a central path for the SDP can be defined as:

$$\begin{aligned} \hat{x}(\mu) &= \operatorname{argmax} \left\{ \mu \log(K - c^T x) + \sum_{j=1}^{q} \log\det A^{(j)}(x) : x \in \mathcal{R} \right\} \\ &= \operatorname{argmin} \left\{ \mu \log(K - c^T x)^{-1} + \sum_{j=1}^{q} \log\det A^{(j)}(x)^{-1} : x \in \mathcal{R} \right\}. \end{aligned}$$

Hence, if follows that $\hat{x}(\mu)$ is a weighted analytic center defined by $A^{(q+1)}(x) := K - c^T x > 0$ and the other q LMI constraints with weight $\omega = (1, \ldots, 1, \mu)$ (see 1.2). As $\mu \to \infty$, the force from the 'cut' $K - c^T x > 0$ pushes the point $\hat{x}(\mu)$ to the point on the boundary of $\mathcal{R}$ where $c^T x$ is minimized. The definition of weighted analytic center (1.2) generalizes the central path. The definition has the added advantage that it can be used to define the concept of repelling path and repelling limits of LMI constraints.

Let $J \subseteq \{1, 2, \ldots, q\}$ and let ω be a weight vector in $\mathbb{R}^q_+$. For $\mu > 0$, we define the ω-*repelling path* associated with the J constraints by

$$x^{(J)}_\omega(\mu) = \operatorname{argmin} \left\{ \sum_{j=1, j\in J}^{q} \mu\omega_j \log\det A^{(j)}(x)^{-1} + \sum_{j=1, j\notin J}^{q} \omega_j \log\det A^{(j)}(x)^{-1} : x \in \mathcal{R} \right\}.$$

Note that if $J = \{1, 2, \ldots, q\} - I$, then $x^{(J)}_\omega(\mu) = x^{(I)}_\omega(1/\mu)$.

The path $x^{(J)}_\omega(\mu)$ is given by the unique optimal solution of the *maxdet* problem

$$\min \sum_{j=1, j\in J}^{q} \mu\omega_j \log\det A^{(j)}(x)^{-1} + \sum_{j=1, j\notin J}^{q} \omega_j \log\det A^{(j)}(x)^{-1}$$

$$\text{s.t. } A^{(j)}(x) := A^{(j)}_0 + \sum_{i=1}^{n} x_i A^{(j)}_i \succ 0, \ \ j = 1, 2, \ldots, q$$

In other words, $x^{(J)}_\omega = x_{ac}(\omega^J(\mu))$, where

$$\omega^J(\mu)_j = \begin{cases} \mu\omega_j & \text{if } j \in J \\ \omega_j & \text{if } j \notin J \end{cases}.$$

The theory of Maxdet optimization is studied in [26].

Corollary 4.4. *The repelling path $g : \mathbb{R}_+ \longrightarrow \mathcal{R}$ defined by $g(\mu) = x^{(J)}_\omega(\mu)$ is analytic in μ.*

Proof. This follows from Corollary 4.1. □

The limit $\lim\limits_{\mu\to\infty} x^{(J)}_\omega(\mu)$ is called the ω-*repelling limit* associated with the J constraints. It is interesting to note that a repelling limit can be an interior point of $\mathcal{R}$ and/or an interior point of $\mathcal{W}$, or neither.

5. COMPUTING THE BOUNDARY OF THE REGION OF WEIGHTED ANALYTIC CENTERS $\mathcal{W}$ USING REPELLING LIMITS

In this section we use the concept of repelling limits defined in the previous section to determine the boundary of the region of weighted analytic centers $\partial\mathcal{W}$. We also use a modification of Newton's method and a new concept of the *frame of* $\mathcal{W}$. This is called the WF algorithm, since it can be used to compute both the $\mathcal{W}$ and the Frame of $\mathcal{W}$.

In our WF algorithm, μ is fixed and $x^{(J)}_\omega(\mu)$ is computed for many random choices of ω and J, where $|J| = n$ and $J \subseteq \{1, 2, \ldots, q\}$ has size n. We show that every repelling limit with $|J| = n$ is either a boundary point of $\mathcal{W}$ or a *frame point* (or both). When μ is large, typically 1000 or 10000, each point $x^{(J)}_\omega(\mu)$ approximates a repelling limit. If $q = n + 1$, these repelling limits give the boundary of the region of weighted analytic centers in the examples we have studied. If $q > n + 1$, we observe frame points in the interior of $\mathcal{W}$ as well as points in $\partial\mathcal{W}$.

The WF algorithm uses Newton's method to find the weighted analytic centers. Newton's method is problematic when the weighted analytic center is near the boundary of $\mathcal{R}$. In particular, the Newton step $x_0 \to x_0 + s$ may take the new point out of $\mathcal{R}$. If this happens, our algorithm uses a step size h such that $x_0 \to x_0 + hs$ is half way to the boundary of $\mathcal{R}$ in the direction of s. This prevents the iterates from leaving $\mathcal{R}$. As an alternative, Newton's method with backtracking could be used. Theorem 5.1 and its corollary give our method for determining the distance to the boundary of $\mathcal{R}$.

Theorem 5.1. *Let s be a nonzero vector in $\mathbb{R}^n$, and $A(x_0) \succ 0$, where $A(x) = A_0 + \sum_{i=1}^n A_i x_i$ is symmetric. Choose a square matrix L such that $A(x_0) = LL^T$. Let $\lambda_{\max}$ be the maximum eigenvalue of the symmetric matrix*

$$B = -L^{-1}\left[\sum_{i=1}^n s_i A_i\right](L^{-1})^T.$$

(a) *If $\lambda_{\max} > 0$, then $A(x_0 + \sigma s) \succ 0$ for all positive $\sigma < 1/\lambda_{\max}$, and $A(x_0 + \sigma s) \not\succ 0$ for all $\sigma \geq 1/\lambda_{\max}$.*

(b) *If $\lambda_{\max} \leq 0$, then $A(x_0 + \sigma s) \succ 0$ for all $\sigma > 0$.*

Proof. The matrix $A(x_0)$ is positive definite, and the region where $A(x) \succ 0$ is convex, so $A(x_0 + \sigma s) \succ 0$ for all σ in the maximal interval including 0 where $\det A(x_0 + \sigma s) > 0$, and $A(x_0 + \sigma s) \not\succ 0$ outside this interval. For simplicity, we only consider $\sigma > 0$. Now, L is nonsingular, since $\det L = \sqrt{\det A(x_0)} > 0$. Furthermore, $(L^{-1})^T = (L^T)^{-1}$ for any nonsingular matrix, so $L^{-1}A(x_0)(L^{-1})^T = I$. Now, for any $\sigma \neq 0$,

$$\begin{aligned}
\det[A(x_0 + \sigma s)] = 0 &\Leftrightarrow \det\left[A(x_0) + \sigma \sum_{i=1}^n s_i A_i\right] = 0 \\
&\Leftrightarrow \det\left[\frac{1}{\sigma} I + L^{-1}\left[\sum_{i=1}^n s_i A_i\right](L^{-1})^T\right] = 0 \\
&\Leftrightarrow \det\left[\frac{1}{\sigma} I - B\right] = 0 \\
&\Leftrightarrow \frac{1}{\sigma} \text{ is an eigenvalue of } B. \qquad (5.1)
\end{aligned}$$

(a) Let $\lambda_{\max}$ be the largest eigenvalue of B. If $\lambda_{\max} > 0$, then $\sigma = 1/\lambda_{\max}$ is the smallest positive σ for which $\det A(x_0+\sigma s) = 0$. This implies that $A(x_0+\sigma s)$ is positive definite for all positive $\sigma < 1/\lambda_{\max}$, and $A(x_0 + \sigma s)$ is not positive definite for all $\sigma \geq 1/\lambda_{\max}$.

(b) If $\lambda_{\max} \leq 0$, then B has no positive eigenvalues and $A(x_0 + \sigma s) \succ 0$ for all $\sigma > 0$.

□

Corollary 5.2. *Let $x_0 \in \mathcal{R}$, $s \in \mathbb{R}^n$, $s \neq 0$. Define $\lambda_{\max}^{(j)}$ for each constraint $A^{(j)}(x) \succ 0$, as in Theorem 5.1. Then $x_0 + \sigma s$, for $\sigma > 0$, is on the boundary of $\mathcal{R}$ if and only if*

$$\sigma = \min\left\{1/\lambda_{\max}^{(j)} : 1 \leq j \leq q \text{ and } \lambda_{\max}^{(j)} > 0\right\}. \qquad (5.2)$$

Proof. By Theorem 5.1, all of the $A^{(j)}(x)$ are positive definite if $x = x_0 + ts$ and $0 < t < \sigma$, and at least one of the $A^{(j)}(x_0 + \sigma s)$ is not positive definite. □

In our Newton-based WF algorithm, if the Newton step $x_0 \mapsto x_0 + s$ maps the point out of the feasible region, then we take the step $x_0 \mapsto x_0 + (\sigma/2)s$, where σ is found in Corollary 5.2, using the Cholesky factorizations of $A^{(j)}(x_0)$. Thus, the new iterate is half way to the boundary from x_0, in the direction of s.

We now give a series of results, culminating with Theorem 5.6 which characterizes the feasible boundary points in this way; at every feasible boundary point of $\mathcal{W}$ there is a set of positively linearly dependent force vectors of size $\leq n$.

The vector sum of the constraint forces $\mathbf{F}^{(j)}(x)$ can be written as the $n \times q$ matrix $M(x)$ times the column vector ω:

$$\sum_{j=1}^q \omega_j \, \mathbf{F}^{(j)}(x) = M(x)\omega,$$

where the columns of $M(x)$ are the boundary forces $\mathbf{F}^{(j)}(x)$. Using Lemma 2.1 we find that

$$M(x) := \left[\mathbf{F}^{(1)}(x) \ \cdots \ \mathbf{F}^{(q)}(x)\right] = \begin{bmatrix} A^{(1)}(x)^{-1} \bullet A_1^{(1)} & \cdots & A^{(q)}(x)^{-1} \bullet A_1^{(q)} \\ \cdots & \cdots & \cdots \\ A^{(1)}(x)^{-1} \bullet A_n^{(1)} & \cdots & A^{(q)}(x)^{-1} \bullet A_n^{(q)} \end{bmatrix}. \tag{5.3}$$

Theorem 5.3. *If $x^* \in \mathcal{R}$, $x^* \in \partial\mathcal{W}$, and $x^* \notin \mathcal{W}$, then there is a nonzero weight vector which satisfies $M(x^*)\omega = 0$, $\omega \geq 0$, but there are no solutions to $M(x^*)\omega = 0$ for $\omega > 0$.*

Proof. Assume that x^* satisfies the hypotheses of the theorem. Since $x^* \in \partial\mathcal{W}$ there is a sequence of points x^n in $\mathcal{W}$ which converges to x^*. Since $x^n \in \mathcal{W}$ for each $n \in \mathbb{N}$, there is sequence of normalized weight vectors $\omega^n \in \Delta^{q-1}$ such that $x_{ac}(\omega^n) = x^n$. The open simplex Δ^{q-1} is a bounded subset of $\mathbb{R}^q$, so there is a convergent subsequence ω^{n_i}. For simplicity relabel this convergent subsequence as ω^i and call its limit ω^*, so that $\omega^i \to \omega^*$ as $i \to \infty$. Now, $M(x)$ is continuous at each $x \in \mathcal{R}$, and $M(x^i)\omega^i = 0$ for each i, so

$$\lim_{i\to\infty} M(x^i)\omega^i = M\left(\lim_{i\to\infty} x^i\right) \lim_{i\to\infty} \omega^i = M(x^*)\omega^* = 0.$$

Since $x^* \notin \mathcal{W}$, we know that $\omega^* \notin \Delta^{q-1}$. Therefore, ω^* is a boundary point of the open simplex, which implies that $\omega^* \geq 0$, $\omega^* \neq 0$, and $\omega^* \not> 0$. (At least one, but not all, of the components of ω^* is zero.) This satisfies the first part of the conclusion of the theorem. Finally, there is no $\omega > 0$ such that $M(x^*)\omega = 0$, since $x^* \notin \mathcal{W}$. □

The next example shows that the converse of Theorem 5.3 is false.

Example 5.1. Consider the system of 4 LMIs

$$A^{(1)}(x) := 1 + x_1 > 0, \ A^{(2)}(x) := 1 - x_1 > 0, \ A^{(3)}(x) := 1 + x_2 > 0, \text{ and}$$

$$A^{(4)}(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \succ 0.$$

The feasible region $\mathcal{R}$ is the interior of the unit circle, and $\mathcal{W}$ is the upper half of the disk. The set of points which satisfy the hypotheses of Theorem 5.3 is the line segment $\{(x_1, x_2) : -1 \leq x_1 \leq 1, \ x_2 = 0\}$. However, the lower half of the disk (with $x_2 \leq 0$) satisfies the conclusion of the theorem. Hence, the converse of Theorem 5.3 is false.

The important ingredient in this example is that $A^{(1)}(x)$ and $A^{(2)}(x)$ are independent of x_2, not that they are linear. Similar examples can be made with $x \in \mathbb{R}^3$ without using linear constraints, by augmenting a set of LMIs which are independent of x_3.

The following theorem gives a geometric characterization of feasible boundary points of $\mathcal{W}$.

Theorem 5.4. *If $x^* \in \mathcal{R}$ and $x^* \in \partial\mathcal{W}$, then there is a nonempty set $J \subseteq \{1, 2, 3, \ldots, q\}$ and a nonzero vector $s \in \mathbb{R}^n$ such that $s \cdot \mathbf{F}^{(j)}(x^*) = 0$ if $j \in J$, $s \cdot \mathbf{F}^{(j)}(x^*) > 0$ if $j \notin J$, and the set of force vectors $\{\mathbf{F}^{(j)}(x^*) : \ j \in J\}$ is positively linearly dependent.*

Proof. First, assume that $x^* \in \partial\mathcal{W}$ and $x^* \in \mathcal{W}$. Theorem 2.6 implies that the set of force vectors do *not* span $\mathbb{R}^n$, so they must lie in a subspace of dimension less than n. Therefore there is a nonzero vector $s \in \mathbb{R}^n$ such that $s \cdot \mathbf{F}^{(j)}(x^*) = 0$ for all j and we choose $J = \{1, 2, 3, \ldots, q\}$.

Now assume that $x^* \in \partial\mathcal{R}$ and $x^* \in \mathcal{R}$ but $x^* \notin \mathcal{W}$. The hypotheses of Theorem 5.3 hold, and we will apply the alternative theorems of Stiemke and Gordon to the conclusions of the theorem. Stiemke's theorem of the alternative ([22], [14]) says that for any $n \times q$ matrix M, either

$$\begin{array}{c} \exists\, \omega > 0 \text{ such that } M\omega = 0 \\ \text{or } \exists\, s \text{ such that } s^T M \geq 0,\ s^T M \neq 0 \\ \text{but not both.} \end{array} \tag{5.4}$$

Gordon's Alternative Theorem [14] is similar: For any $n \times q$ matrix M, either

$$\begin{array}{c} \exists\, \omega \geq 0 \text{ such that } \omega \neq 0 \text{ and } M\omega = 0 \\ \text{or } \exists\, s \text{ such that } s^T M > 0 \\ \text{but not both.} \end{array} \tag{5.5}$$

Since $x^* \notin \mathcal{W}$, Stiemke's theorem says that there exists $s \in \mathbb{R}^n$ such that $s \cdot \mathbf{F}^{(j)}(x^*) \geq 0$ for all $j \in \{1, 2, \ldots, q\}$, and $s \cdot \mathbf{F}^{(j)}(x^*) > 0$ for some j. However, Gordon's theorem of the alternative says that there is no s such that $s \cdot \mathbf{F}^{(j)}(x^*) > 0$ for all j. Let J be the smallest subset of $\{1, 2, \ldots, q\}$ such that there is an $s \in \mathbb{R}^n$ with the property that $s \cdot \mathbf{F}^{(j)}(x^*) = 0$ if $j \in J$ and $s \cdot \mathbf{F}^{(j)}(x^*) > 0$ if $j \notin J$. Fix s to be one such vector. The set J is nonempty due to Gordon's Theorem.

To complete the proof we must show that the set $\{\mathbf{F}^{(j)}(x^*) :\ j \in J\}$ is positively linearly dependent. Assume, by way of contradiction, that it is not. Then, Stiemke's theorem implies that there is a vector $t \in \mathbb{R}^n$ such that $t \cdot \mathbf{F}^{(j)}(x^*) \geq 0$ for all $j \in J$, and an integer $j^* \in J$ such that $t \cdot \mathbf{F}^{(j^*)}(x^*) > 0$. Using the s defined previously, choose a positive ϵ so that $(s + \epsilon\, t) \cdot \mathbf{F}^{(j)}(x^*) > 0$ for all $j \notin J$. Then $(s + \epsilon\, t) \cdot \mathbf{F}^{(j)}(x^*) = \epsilon\, t \cdot \mathbf{F}^{(j)}(x^*) \geq 0$ for all $j \in J$, and $(s + \epsilon\, t) \cdot \mathbf{F}^{(j^*)}(x^*) > 0$. Since $j^* \in J$, this contradicts the fact that J is minimal. Therefore, $\{\mathbf{F}^{(j)}(x^*) :\ j \in J\}$ is positively linearly dependent, and the theorem is proved. ☐

Remark 5.5. Let us apply Theorem 5.4 to Example 3.4, for which $\mathcal{W}$ is shown in Figure 3.4.

At the feasible boundary point $e = (0, 1)$, the force vectors are $\mathbf{F}^{(1)}(e) = \mathbf{0}$, $\mathbf{F}^{(2)}(e) = 4\mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{F}^{(3)}(e) = \mathbf{e}_1 + \mathbf{e}_2$. (We have chosen a convenient normalization of the force vectors since only their direction matters.) The set from Theorem 5.4 is $J = \{1\}$, and a choice of the vector is $s = \mathbf{e}_1$.

A more typical feasible boundary point of $\mathcal{W}$ is $(1, 5/4)$, where the force vectors are $\mathbf{F}^{(1)}(1, 5/4) = -\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{F}^{(2)}(1, 5/4) = -\mathbf{e}_2$, and $\mathbf{F}^{(3)}(1, 5/4) = \mathbf{e}_1 + \mathbf{e}_2$. Here the set from Theorem 5.4 is $J = \{1, 3\}$, and the vector $s = \mathbf{e}_1 - \mathbf{e}_2$ is unique up to a positive scalar multiple.

The feasible boundary point $d = (4/3, 4/3)$ is a point in $\mathcal{W}$ which is not an interior point. The force vectors are $\mathbf{F}^{(1)}(d) = -\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{F}^{(2)}(d) = -\mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{F}^{(3)}(d) = \mathbf{e}_1 + \mathbf{e}_2$. Here the set from Theorem 5.4 is $J = \{1, 2, 3\}$, and the vector $s = \mathbf{e}_1 - \mathbf{e}_2$ is unique up to a nonzero scalar multiple.

We need one more lemma before we can characterize feasible boundary points of $\mathcal{W}$. Our proof of the lemma is similar to the proof of the Carathéodory's Theorem given in [8].

Lemma 5.6. *Every positively linearly dependent set of vectors in $\mathbb{R}^n$ contains a positively linearly dependent subset of size $n + 1$ or smaller.*

Proof. Assume, by way of contradiction, that a set $\mathcal{S} := \{\mathbf{F}^{(j)} \in \mathbb{R}^n :\ j = 1, 2, 3, \ldots, p\}$, with $p > n + 1$, is positively linearly dependent, and there is no proper subset which is positively linearly dependent. Choose a set of positive weights ω_j such that

$$\sum_{j=1}^{p} \omega_j \mathbf{F}^{(j)} = 0.$$

The set $\mathcal{S}$ is affinely dependent, meaning that there are weights α_j (not all zero) such that

$$\sum_{j=1}^{p} \alpha_j \mathbf{F}^{(j)} = 0 \text{ and } \sum_{j=1}^{p} \alpha_j = 0.$$

This is true since $\{\mathbf{F}^{(j)} - \mathbf{F}^{(p)} : \ j = 1, 2, 3, \ldots, p-1\}$, being a set of vectors in $\mathbb{R}^n$ of size at least $n+1$, is linearly dependent.

Choose such a set of weights α_j. For any $\lambda \in \mathbb{R}$, it follows that

$$\sum_{j=1}^{p} (\omega_j + \lambda\alpha_j)\mathbf{F}^{(j)} = 0.$$

Now, define λ^* so that all of the weights $\omega_j + \lambda^*\alpha_j$ are nonnegative but at least one is zero.

$$\lambda^* = \max\{\lambda : \omega_j + \lambda\alpha_j \geq 0 \text{ for } j = 1, 2, \ldots, p\}.$$

Since at least one of the α_j is negative, λ^* exists. Now, define $J = \{j : \ \omega_j + \lambda^*\alpha_j > 0\}$. By construction, $1 \leq |J| < p$, and the sum

$$\sum_{j \in J} (\omega_j + \lambda^*\alpha_j)\mathbf{F}^{(j)} = 0$$

demonstrates that the set $\{\mathbf{F}^{(j)} \in \mathbb{R}^n : \ j \in J\}$ is positively linearly dependent. This contradicts the assumption that no proper subset of $\mathcal{S}$ is positively linearly dependent. □

Remark 5.7. The "or smaller" is needed in the statement of Lemma 5.6. The positively linearly dependent set of vectors in $\mathbb{R}^2$, $\{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$ has no positively linearly dependent subset of size 3. Theorem 5.6 holds, of course, and $\{\mathbf{e}_1, -\mathbf{e}_1\}$ is a positively linearly dependent subset of size 2.

We define the *frame* of $\mathcal{W}$, denoted by $\mathcal{F} \subseteq \mathcal{R}$, to be the set

(5.6) $$\mathcal{F} := \left\{x : \ \exists \text{ a positively linearly dependent subset of } \{\mathbf{F}^{(j)}(x)\}_{j=1}^{q} \text{ of size } \leq n\right\}.$$

If $x \in \mathcal{F}$, we call x a *frame point*. We now state our **main theorem**. It gives a useful characterization of the boundary points of $\mathcal{W}$.

Theorem 5.8. *Every feasible boundary point of $\mathcal{W}$ is a frame point. That is, if $x^* \in \partial W \cap \mathcal{R}$, then there is a set of n or fewer positively linearly dependent force vectors $\mathbf{F}^{(j)}(x^*)$.*

Proof. Assume that $x^* \in \partial\mathcal{W} \cap \mathcal{R}$. By Theorem 5.4, there is a positively linearly dependent subset of vectors $\{\mathbf{F}^{(j)}(x^*) : \ j \in J\}$ and a nonzero vector $s \in \mathbb{R}^n$ such that $s \cdot \mathbf{F}^{(j)}(x^*) = 0$ if $j \in J$. All of the vectors $\mathbf{F}^{(j)}(x^*)$ with $j \in J$ lie in the $(n-1)$-dimensional subspace perpendicular to s. Therefore, by Theorem 5.6, there is a positively linearly dependent subset of $\{\mathbf{F}^{(j)}(x^*) : \ j \in J\}$ of size $(n-1)+1 = n$ or smaller. Therefore, x^* is a frame point. □

Theorem 5.8 is a significant improvement over Theorem 5.4, which gives no information about the size of the set J. Our main result (Theorem 5.8) motivates the WF algorithm. The implementation uses the following corollary.

Corollary 5.9. *Every repelling limit $\lim_{\mu\to\infty} x_\omega^{(J)}(\mu)$, where J is of size n, is either a frame point or an infeasible boundary point of $\mathcal{W}$.*

Proof. Let x^* be a repelling limit $\lim_{\mu\to\infty} x_\omega^{(J)}(\mu)$ for $|J| = n$. If x^* is in $\mathcal{R}$, then there exists n positively linearly independent force vectors $\mathbf{F}^{(j)}(x^*)$ at x^*. So, x^* is a frame point. If $x^* \in \partial\mathcal{R}$, then $x^* \notin W$. By Theorem 5.8, $x^* \in \partial W$. □

The WF algorithm approximates the repelling limits $\lim_{\mu\to\infty} x_\omega^{(J)}(\mu)$, where J is a randomly chosen set of size n, by choosing large fixed μ. This approximates the frame points at which there exist n force vectors which are positively linearly dependent. (The frame points where fewer than n force vectors are positively linearly dependent can be approximated with $|J| = n$ when one or more random weights are small.)

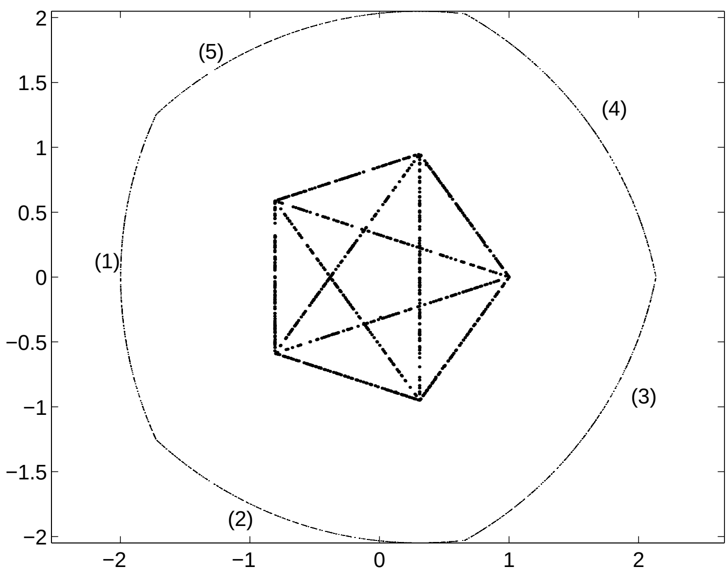

Figure 5.1: The frame of $\mathcal{W}$ for Example 3.3 with $q = 5$ is composed of the line segments joining the five corner points of the pentagon. The figure shows $N = 1,000$ points which approximate the frame, obtained with the WF algorithm using $\mu = 10,000$. This gives a better picture of $\partial\mathcal{W}$ than Figure 3.3, where 10,000 points are plotted.

Figures 5.1, 5.2, and 5.3 show the result of the WF algorithm applied to Examples 3.3, 3.4, and 5.1 (respectively), with large μ. Note that the frame points *and* the infeasible boundary points are found by the algorithm. Figure 5.3 shows that the converse of Corollary 5.9 is false: there are frame points, e.g. $(0, -0.5)$ which are not repelling limits.

As stated earlier, the WF algorithm, with μ large, approximates the frame of $\mathcal{W}$. To understand why the algorithm also approximates infeasible boundary points, assume for simplicity that x^* is an infeasible boundary point of $\mathcal{W}$ at which exactly one of the matrices, call it $A^{(b)}(x^*)$, is singular. When the WF algorithm chooses J with $b \notin J$, all but $n+1$ force vectors are negligible for x near x^*:

$$\sum_{j\notin J} \omega_j \mathbf{F}^{(j)}(x) + \sum_{j\in J} \mu\omega_j \mathbf{F}^{(j)}(x) \approx \omega_b \mathbf{F}^{(b)}(x) + \sum_{j\in J} \mu\omega_j \mathbf{F}^{(j)}(x).$$

Thus, the WF algorithm with μ large approximates points where the $n+1$ “large” forces, $\mathbf{F}^{(b)}(x)$ and $\mu\mathbf{F}^{(j)}(x)$ with $j \in J$, are positively linearly dependent. Lemma 5.6 implies that neglecting the other force vectors will not prevent us from finding x^* in this way.

A similar argument holds if the infeasible boundary point is a “corner” point of $\mathcal{R}$, where two (or more) of the matrices $A^{(j)}(x^*)$ are singular. An example is point b in Figure 5.2. We suspect that all infeasible boundary points of $\mathcal{W}$ are approximated by the WF algorithm in this way.

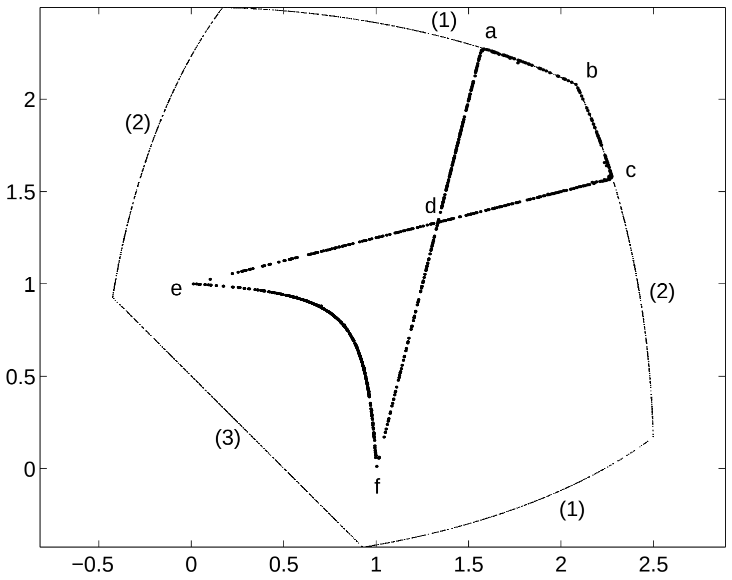

Figure 5.2: The boundary of $\mathcal{W}$ in Example 3.4 is approximated using the WF algorithm with $N = 1,000$ points and $\mu = 10,000$. Since $q = n + 1$, this example has no frame points which are not on the boundary of $\mathcal{W}$. These 1,000 points should be compared with the 10,000 points plotted in Figure 3.4.

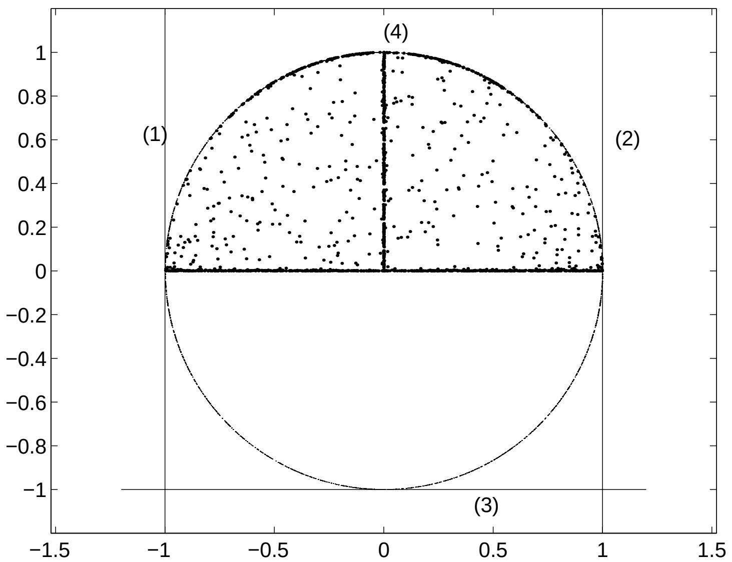

Figure 5.3: The output of the WF Algorithm applied to Example 5.1, showing $N = 2,000$ points using $\mu = 1,000$. In this pathological example, W is the line segment joining $(-1,0)$ to $(1,0)$ and the frame of $\mathcal{W}$ is the whole feasible disk, since $\mathbf{F}^{(1)}(x)$ and $\mathbf{F}^{(2)}(x)$ are positively linearly dependent at all x. Frame points which are not in the closure of $\mathcal{W}$ are not found by our WF Algorithm, since these are not repelling limits.

6. CONCLUSION

We studied the boundary of the region of weighted analytic centers $\mathcal{W}$ for linear matrix inequality constraints. This is important because many interior point methods in semidefinite programming are based on weighted analytic centers. We gave many examples that illustrated the topological properties of $\mathcal{W}$. We showed that the region of weighted analytic centers is not necessarily convex in $\mathbb{R}^n$, for all $n > 1$. We also showed that $\mathcal{W}$ is not open if $q \leq n$, and $\mathcal{W}$ is not necessarily open if $q > n$.

We extended the notion of repelling paths and repelling limits from linear programming to semidefinite programming . We gave a characterization of the boundary points of $\mathcal{W}$ and introduced the new concept of the frame of $\mathcal{W}$. We show that feasible boundary points of $\mathcal{W}$ are frame points of $\mathcal{W}$. We developed the WF algorithm, which can compute either $\mathcal{W}$ or the boundary of $\mathcal{W}$ in terms of repelling limits, based on a modification of Newton's method.

There are many directions that our present work can be extended. To avoid the problem of unbounded derivatives at the boundary or $\mathcal{R}$, we can choose positive weights ν_j and solve

$$\sum_{j=1}^{q} \nu_j \nabla_i |A^{(j)}(x)| = 0 \text{ for } i = 1, 2, \ldots, n. \tag{6.1}$$

If Newton's method converges to a solution, and if the solution is in $\mathcal{R}$, then it is in $\mathcal{W}$. We have had success in preliminary investigations with this method, but Newton's method does not always converge, or it converges to an infeasible point.

The infeasible boundary points of $\mathcal{W}$ can be computed using a modification of Lagrange multipliers. For example a point on the b^{th} boundary component of $\mathcal{R}$ can be found by solving the $n+1$ equations in the $n+1$ variables λ, x:

$$|A^{(b)}(x)| = 0,$$

$$\lambda \nabla |A^{(b)}(x)| + \sum_{j \neq b} \omega_j \frac{\nabla |A^{(j)}(x)|}{|A^{(j)}(x)|} = 0,$$

where the weights ω_j are chosen randomly for $j \neq b$. Spurious solutions to this system abound. One must check that $\lambda > 0$ and that x is indeed on the b^{th} boundary component of $\mathcal{R}$.

Preliminary investigations show that points near the intersection of two boundaries are rarely found by this method. This problem is alleviated if we combine the Lagrange multiplier method with the polynomial method described in system (6.1). Choose an integer b at random, and choose the $q-1$ weights ν_j, $j \neq b$ at random and solve

$$|A^{(b)}(x)| = 0$$

$$\lambda \nabla |A^{(b)}(x)| + \sum_{j \neq b} \nu_j \nabla |A^{(j)}(x)| = 0$$

for the $n+1$ variables λ and x. As before, Newton's method does not always converge, and it sometimes converges to points that are not on the boundary of $\mathcal{R}$.

The definition of weighted analytic center led to the concepts of *positive linear dependence* and *positive convex hull*. Another way that this work can be extended is to explore different definitions of the weighted analytic center (for example, replacing $\omega_j > 0$ with $\omega_j \geq 0$).

APPENDIX: THE WF ALGORITHM (WFA)

Algorithm to plot points in $\mathcal{W}$ ($\mu = 1$) or Frame and boundary points of $\mathcal{W}$ ($\mu \gg 1$)

Input: Any feasible point x^* of q LMIs in $\mathbb{R}^n$, number of points to plot `N`, stoping conditions for Newton's method `TOL` and `maxIts`, and a parameter $\mu = 1$ or $\mu \gg 1$.
Repeat
  If $\mu > 1$ and $q > n$, choose a random set $J \subseteq \{1, 2, \ldots, q\}$ of size n
  Choose q numbers u_j randomly and independently from $U(0, 1)$
  Set $\omega_j = -\log(u_j)$ (See Lemma 2.5.)
  If $\mu > 1$ and $q > n$, set $\omega_j = \mu * \omega_j$ for all $j \in J$
  Set $x_0 = x^*$, $num = 0$ and $k = 0$
  Repeat
    Compute $H_\omega(x_k)$ and $\nabla\phi_\omega(x_k)$
    Solve the linear system $H_\omega(x_k)s = -\nabla\phi_\omega(x_k)$ for the Newton step s
    If $x_k + s$ is infeasible,
      Calculate $\sigma > 0$ such that $x_k + \sigma s \in \partial\mathcal{W}$, using Corollary 5.2
      Set $h = 0.5\sigma$
    Else set $h = 1$
    Update $x_{k+1} := x_k + hs;\ k := k + 1$
  Until $\sqrt{s^T s} \leq$ `TOL` or $k >$ `maxIts`
  If $\sqrt{s^T s} \leq$ `TOL` plot $x_k \approx x_\omega^{(J)}(\mu)$
  Set $num = num + 1$
Until $num =$ `N`

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