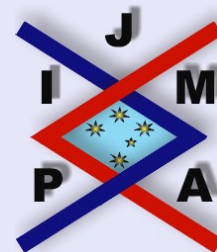


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Abstract

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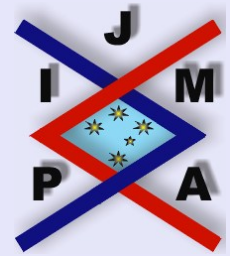


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## Abstract

Considering two given real parameters  $\alpha, \beta$  which satisfy the condition  $0 \leq \alpha \leq \beta$ , D.D. Stancu ([11]) constructed and studied the linear positive operators  $P_m^{(\alpha, \beta)} : C([0, 1]) \rightarrow C([0, 1])$ , defined for any  $f \in C([0, 1])$  and any  $m \in \mathbb{N}$  by

$$\left( P_m^{(\alpha, \beta)} f \right) (x) = \sum_{k=0}^m p_{mk}(x) f \left( \frac{k + \alpha}{m + \beta} \right).$$

In this paper, we are dealing with the Kantorovich form of the above operators. We construct the linear positive operators  $K_m^{(\alpha, \beta)} : L_1([0, 1]) \rightarrow C([0, 1])$ , defined for any  $f \in L_1([0, 1])$  and any  $m \in \mathbb{N}$  by

$$\left( K_m^{(\alpha, \beta)} f \right) (x) = (m + \beta + 1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s) ds$$

and we study some approximation properties of the sequence  $\left\{ K_m^{(\alpha, \beta)} \right\}_{m \in \mathbb{N}}$ .

**2000 Mathematics Subject Classification:** 41A36, 41A25

**Key words:** Linear positive operators, Bernstein operator, Kantorovich operator, Stancu operator, First order modulus of smoothness, Shisha-Mond theorem.

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# 1. Preliminaries

Starting with two given real parameters  $\alpha, \beta$  satisfying the conditions  $0 \leq \alpha \leq \beta$  in 1968, D.D. Stancu (see [11]) constructed and studied the linear positive operators  $P_m^{(\alpha, \beta)} : C([0, 1]) \rightarrow C([0, 1])$  defined for any  $f \in C([0, 1])$  and any  $m \in \mathbb{N}$  by

$$(1.1) \quad (P_m^{(\alpha, \beta)} f) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{k + \alpha}{m + \beta}\right),$$

where  $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$  are the fundamental Bernstein polynomials ([5]).

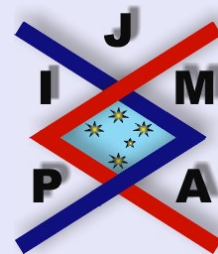
The operators (1.1) are known in mathematical literature as "the operators of D.D. Stancu" (see ([2])).

Note that for  $\alpha = \beta = 0$ , the operator  $P_m^{(0,0)}$  is the classical Bernstein operator  $B_m$  ([5]).

In 1930, L.V. Kantorovich constructed and studied the linear positive operators  $K_m : L_1([0, 1]) \rightarrow C([0, 1])$  defined for any  $f \in L_1([0, 1])$  and any non-negative integer  $m$  by

$$(1.2) \quad (K_m f)(x) = (m + 1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s) ds.$$

The operators (1.2) are known as the Kantorovich operators. These operators are obtained from the classical Bernstein operators (1.1), replacing there the value  $f(k/m)$  of the approximated function by the integral of  $f$  in a neighborhood of  $k/m$ .



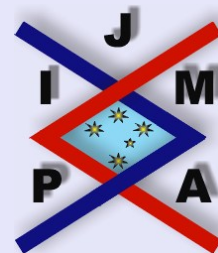
Following the ideas of L.V. Kantorovich ([7]), let us consider the operators  $K_m^{(\alpha, \beta)} : L_1([0, 1]) \rightarrow C([0, 1])$ , defined for any  $f \in C([0, 1])$  and any  $m \in \mathbb{N}$  by

$$(1.3) \quad (K_m^{(\alpha, \beta)} f)(x) = (m + \beta + 1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s) ds$$

obtained from the Stancu type operators (1.1).

Section 2 provides some interesting approximation properties of operators (1.3), called "Kantorovich-Stancu type operators" because they are obtained starting from the Stancu type operators (1.1) following Kantorovich's ideas (see also G.G. Lorentz [9]).

A convergence theorem for the sequence  $\left\{ K_m^{(\alpha, \beta)} f \right\}_{m \in \mathbb{N}}$  is proved and the rate of convergence under some assumptions on the approximated function  $f$  is evaluated.




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## 2. Main Results

**Lemma 2.1.** *The Kantorovich-Stancu type operators (1.3) are linear and positive.*

*Proof.* The assertion follows from definition (1.3). □

In what follows we will denote by  $e_k(s) = s^k, k \in \mathbb{N}$ , the test functions.

**Lemma 2.2.** *The operators (1.3) verify*

$$(2.1) \quad (K_m^{(\alpha, \beta)} e_0)(x) = 1,$$

$$(2.2) \quad (K_m^{(\alpha, \beta)} e_1)(x) = \frac{m}{m + \beta + 1}x + \frac{\alpha}{m + \beta + 1} + \frac{m + \beta}{2(m + \beta + 1)^2},$$

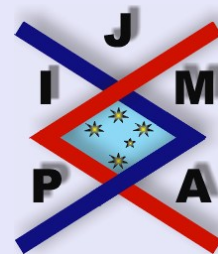
$$(2.3) \quad (K_m^{(\alpha, \beta)} e_2)(x) = \frac{1}{(m + \beta + 1)^2} \left\{ m^2 x^2 + mx(1 - x) + \frac{2\alpha m^2}{m + \beta} + \frac{\alpha^2(3m + \beta)}{m + \beta} \right\} + \frac{1}{(m + \beta + 1)^2} \{mk + \alpha\} + \frac{1}{3(m + \beta + 1)^2}$$

for any  $x \in [0, 1]$ .

*Proof.* It is well known (see [11]) that the Stancu type operators (1.1) satisfy

$$(P_m^{(\alpha, \beta)} e_0)(x) = 1$$

$$(P_m^{(\alpha, \beta)} e_1)(x) = \frac{m}{m + \beta}x + \frac{\alpha}{m + \beta}$$




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$$(P_m^{(\alpha,\beta)} e_2)(x) = \frac{1}{(m+\beta)^2} \left\{ m^2 x^2 + mx(1-x) + 2 \frac{\alpha m^2}{m+\beta} x + \frac{3\alpha^2 m}{m+\beta} \right\}$$

□

Next we apply the definition (2.1).

**Lemma 2.3.** *The operators (1.3) satisfy*

$$\begin{aligned} (2.4) \quad K_m^{(\alpha,\beta)}((e_1 - x)^2; x) &= \frac{(\beta + 1)^2}{(m + \beta + 1)^2} x^2 + \frac{m}{(m + \beta + 1)^2} x(1 - x) \\ &\quad + \frac{m}{(m + \beta + 1)^2(m + \beta)} \{m + 2\alpha(m - \beta - 1)\}x \\ &\quad + \frac{3\alpha^2(3m + \beta) + (m + \beta)(1 - 3m - 3\beta)}{3(m + \beta)(m + \beta + 1)^2} \end{aligned}$$

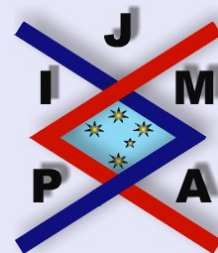
for any  $x \in [0, 1]$ .

*Proof.* From the linearity of  $K_m^{(\alpha,\beta)}$ , we get

$$\begin{aligned} K_m^{(\alpha,\beta)}((e_1 - x)^2; x) &= (K_m^{(\alpha,\beta)} e_2)(x) - 2x K_m^{(\alpha,\beta)}(e_1; x) + x^2 (K_m^{(\alpha,\beta)} e_0)(x) \end{aligned}$$

□

Next, we apply Lemma 2.2.



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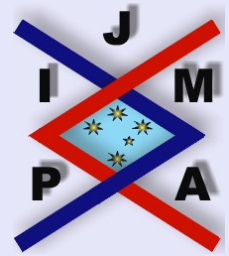


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**Theorem 2.4.** The sequence  $\left\{ K_m^{(\alpha, \beta)} f \right\}_{m \in \mathbb{N}}$  converges to  $f$ , uniformly on  $[0, 1]$ , for any  $f \in L_1([0, 1])$ .

*Proof.* Using Lemma 2.3, we get

$$\lim_{m \rightarrow \infty} K_m^{(\alpha, \beta)}((e_1 - x)^2; x) = 0$$

uniformly on  $[0, 1]$ . We can then apply the well known Bohman-Korovkin Theorem (see [6] and [8]) to obtain the desired result.  $\square$

Next, we deal with the rate of convergence for the sequence  $\left\{ K_m^{(\alpha, \beta)} f \right\}_{m \in \mathbb{N}}$ , under some assumptions on the approximated function  $f$ . In this sense, the first order modulus of smoothness will be used.

Let us recall that if  $I \subseteq \mathbb{R}$  is an interval of the real axis and  $f$  is a real valued function defined on  $I$  and bounded on this interval, the first order modulus of smoothness for  $f$  is the function  $\omega_1 : [0, +\infty) \rightarrow \mathbb{R}$ , defined for any  $\delta \geq 0$  by

$$(2.5) \quad \omega_1(f; \delta) = \sup \{ |f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta \}.$$

For more details, see for example [1].

**Theorem 2.5.** For any  $f \in L_1([0, 1])$ , any  $\alpha, \beta \geq 0$  satisfying the condition  $\alpha \leq \beta$  and each  $x \in [0, 1]$  the Kantorovich-Stancu type operators (1.3) satisfy

$$(2.6) \quad \left| (K_m^{(\alpha, \beta)} f)(x) - f(x) \right| \leq 2\omega_1 \left( f; \sqrt{\delta_m^{(\alpha, \beta)}(x)} \right),$$

where

$$(2.7) \quad \delta_m^{(\alpha, \beta)}(x) = K_m^{(\alpha, \beta)}((e_1 - x)^2; x)$$

*Proof.* From Lemma 2.2 follows

$$(2.8) \quad \left| (K_m^{(\alpha,\beta)} f)(x) - f(x) \right| \leq (m + \beta + 1) \sum_{k=0}^{m+p} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} |f(s) - f(x)| ds$$

On the other hand

$$|f(s) - f(x)| \leq \omega_1(f; |s - x|) \leq (1 + \delta^{-2}(s - x)^2) \omega_1(f; \delta).$$

For  $|s - x| < \delta$ , the last increase is clear. For  $|s - x| \geq \delta$ , we use the following properties

$$\omega_1(f; \lambda\delta) \leq (1 + \lambda) \omega_1(f; \delta) \leq (1 + \lambda^2) \omega_1(f; \delta),$$

where we choose  $\lambda = \delta^{-1} \cdot |s - x|$ .

This way, after some elementary transformation, (2.8) implies

$$(2.9) \quad \left| (K_m^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \left\{ (K_m^{(\alpha,\beta)} e_0)(x) + \delta^{-2} K_m^{(\alpha,\beta)}((e_1 - x)^2; x) \right\} \omega_1(f; \delta)$$

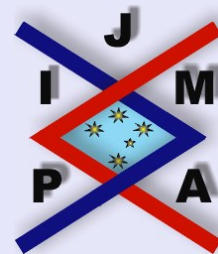
for any  $\delta > 0$  and each  $x \in [0, 1]$ .

Using next Lemma 2.2 and Lemma 2.3, from (2.9) one obtains

$$(2.10) \quad \left| (K_m^{(\alpha,\beta)} f)(x) - f(x) \right| \leq (1 + \delta^{-2} \delta_m^{(\alpha,\beta)}(x)) \omega_1(f; \delta)$$

for any  $\delta \geq 0$  and each  $x \in [0, 1]$ .

Taking into account Lemma 2.1, it follows that  $\delta_m^{(\alpha,\beta)}(x) \geq 0$  for each  $x \in [0, 1]$ . Consequently, we can take  $\delta := \delta_m^{(\alpha,\beta)}(x)$  in (2.9), arriving at the desired result.  $\square$



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**Theorem 2.6.** For any  $f \in L_1([0, 1])$  and any  $x \in [0, 1]$  the following

$$(2.11) \quad |(K_m^{(\alpha, \beta)} f)(x) - f(x)| \leq 2\omega_1 \left( f; \sqrt{\delta_m^{(\alpha, \beta)}}, 1 \right)$$

holds, where

$$(2.12) \quad \delta_{m,1}^{(\alpha, \beta)} = \frac{(\beta + 1)^2}{(m + \beta + 1)^2} + \frac{m^2(2\alpha + 1)}{(m + \beta)(m + \beta + 1)^2} + \frac{m}{4(m + \beta + 1)^2} + \frac{3\alpha^2(3m + \beta) + (m + \beta)(1 - 3m - 3\beta)}{3(m + \beta)(m + \beta + 1)^2}.$$

*Proof.* For any  $x \in [0, 1]$ , the inequality

$$K_m^{(\alpha, \beta)}((e_1 - x)^2; x) \leq \delta_{m,1}^{(\alpha, \beta)}$$

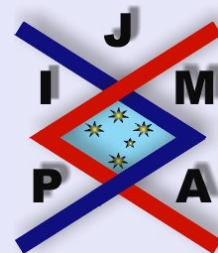
holds. Consequently, applying Theorem 2.5 we get (2.11). □

**Remark 2.1.** Theorem 2.5 gives us the order of local approximation (in each point  $x \in [0, 1]$ ), while Theorem 2.6 contains an evaluation for the global order of approximation (in any point  $x \in [0, 1]$ ).

Because the maximum of  $\delta_m^{(\alpha, \beta)}(x)$  from (2.6) depends on the relations between  $\alpha$  and  $\beta$ , it follows that it can be refined further.

Taking into account the inclusion  $C([0, 1]) \subset L_1([0, 1])$ , as consequences of Theorem 2.5 and Theorem 2.6, follows the following two results.

**Corollary 2.7.** For any  $f \in C([0, 1])$ , any  $\alpha, \beta \geq 0$  satisfying the condition  $\alpha \leq \beta$  and each  $x \in [0, 1]$ , the inequality (2.6) holds.



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**Corollary 2.8.** For any  $f \in C([0, 1])$ , any  $\alpha, \beta \geq 0$  satisfying the condition  $\alpha \leq \beta$  and any  $x \in [0, 1]$ , the inequality (2.11) holds.

Further, we estimate the rate of convergence for smooth functions.

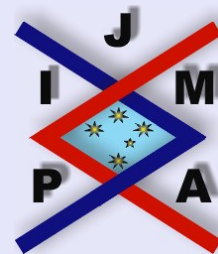
**Theorem 2.9.** For any  $f \in C^1([0, 1])$  and each  $x \in [0, 1]$  the operators (1.3) verify

$$(2.13) \quad \left| (K_m^{(\alpha, \beta)} f)(x) - f(x) \right| \leq |f'(x)| \cdot \left| \frac{m + \beta}{2(m + \beta + 1)^2} - \frac{\beta + 1}{(m + \beta + 1)^2} x \right| + 2\sqrt{2\delta_m^{(\alpha, \beta)}(x)\omega_1\left(f'; \sqrt{\delta_m^{(\alpha, \beta)}(x)}\right)},$$

where  $\delta_m^{(\alpha, \beta)}(x)$  is given in (2.7).

*Proof.* Applying a well known result due to O. Shisha and B. Mond (see [10]), it follows that

$$(2.14) \quad \left| (K_m^{(\alpha, \beta)} f)(x) - f(x) \right| \leq |f(x)| \cdot \left| (K_m^{(\alpha, \beta)} e_0)(x) - 1 \right| + |f'(x)| \cdot \left| (K_m^{(\alpha, \beta)} e_1)(x) - x (K_m^{(\alpha, \beta)} e_0)(x) \right| + \sqrt{K_m^{(\alpha, \beta)}((e_1 - x)^2; x)} \times \left\{ \sqrt{(K_m^{(\alpha, \beta)} e_0)(x)} + \delta^{-1} \sqrt{K_m^{(\alpha, \beta)}((e_1 - x)^2; x)} \right\} \omega_1(f'; \delta).$$



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From (2.14), using Lemma 2.2 and Lemma 2.3, we get

$$(2.15) \quad \left| (K_m^{(\alpha, \beta)} f)(x) - f(x) \right| \leq |f'(x)| \cdot \left| \frac{m + \beta}{(m + \beta + 1)^2} - \frac{\beta + 1}{(m + \beta + 1)^2 x} \right| + \sqrt{\delta_m^{(\alpha, \beta)}(x)} \left\{ 1 + \delta^{-1} \sqrt{\delta_m^{(\alpha, \beta)}(x)} \right\} \omega_1(f'; \delta).$$

Choosing  $\delta = \sqrt{\delta_m^{(\alpha, \beta)}(x)}$  in (2.15), we arrive at the desired result.  $\square$

**Theorem 2.10.** For any  $f \in C^1([0, 1])$  and any  $x \in [0, 1]$  the operators (1.3) verify

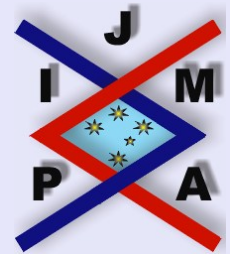
$$(2.16) \quad \left| (K_m^{(\alpha, \beta)} f)(x) - f(x) \right| \leq \frac{m + \beta}{(m + \beta + 1)^2} M_1 + 2\sqrt{\delta} \omega_1(f'; \sqrt{\delta}),$$

where

$$M_1 = \max_{x \in [0, 1]} |f'(x)|, \quad \delta = \max_{x \in [0, 1]} \delta_m^{(\alpha, \beta)}(x).$$

*Proof.* The assertion follows from Theorem 2.9.  $\square$

**Remark 2.2.** Because  $\delta$  depends on the relation between  $\alpha$  and  $\beta$ , (2.16) can be further refined, following the ideas of D.D. Stancu [11, 12].



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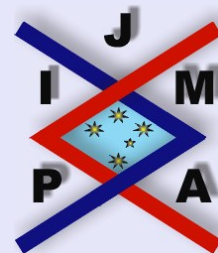
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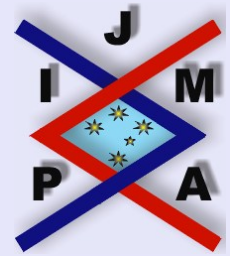
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