ON THE WEIGHTED OSTROWSKI INEQUALITY

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Abstract: On utilising an identity from [5], some weighted Ostrowski type inequalities are established.

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1. Introduction

In [5], the authors obtained the following generalisation of the weighted *Montgomery identity*:

$$f(x)=\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt + \frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt, \tag{1.1}$$

where $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function, $\varphi:[0,1]\to\mathbb{R}$ is a differentiable function with $\varphi(0)=0$, $\varphi(1)\neq 0$ and $w:[a,b]\to[0,\infty)$ is a probability density function such that the weighed *Peano kernel*

$$P_{w,\varphi}(x,t):=\begin{cases}\varphi\left(\int_a^t w(s)\,ds\right), & a\le t\le x,\\ \varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1), & x<t\le b,\end{cases} \tag{1.2}$$

is integrable for any $x\in[a,b]$.

If $\varphi(t)=t$, then (1.1) reduces to the weighted Montgomery identity obtained by Pečarić in [21]:

$$f(x)=\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt, \tag{1.3}$$

where the weighted Peano kernel P_w is

$$P_w(x,t):=\begin{cases}\int_a^t w(s)\,ds, & a\le t\le x,\\ -\int_t^b w(s)\,ds, & x<t\le b.\end{cases} \tag{1.4}$$

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Finally, the uniform distribution is used to provide the Montgomery identity [17, p. 565]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t)\,f'(t)\,dt, \tag{1.5}$$

with

$$P(x,t) := \begin{cases} \frac{t-a}{b-a} & \text{if } a \le t \le x, \\ \frac{t-b}{b-a} & \text{if } x < t \le b, \end{cases}$$

that has been extensively used to obtain Ostrowski type results, see for instance the research papers [3] – [6], [7] – [16], [19] – [20], [22] and the book [15].

In the same paper [5], on introducing the generalised Čebyšev functional,

$$\begin{aligned} T_{\varphi}(w,f,g) := {} & \int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) f(x)\,g(x)\,dx \\ & - \frac{1}{\varphi(1)}\left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) f(x)\,dx\right] \\ & \times \left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) g(x)\,dx\right], \end{aligned} \tag{1.6}$$

the authors obtained the representation:

$$\begin{aligned} T_{\varphi}(w,f,g) = {} & \frac{1}{\varphi^2(1)}\int_a^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) \\ & \times \left[\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt\right]\left[\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt\right] dx \end{aligned} \tag{1.7}$$

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and used it to obtain an upper bound for the absolute value of the Čebyšev functional in the case where $f', g', \varphi' \in L_\infty[a,b]$. This bound can be stated as:

$$|T_\varphi(w,f,g)| \leq \frac{1}{\varphi^2(1)} \|f'\|_\infty \|g'\|_\infty \|\varphi'\|_\infty \int_a^b w(x) H^2(x)\, dx, \tag{1.8}$$

where $H(x) := \int_a^b |P_{w,\varphi}(x,t)|\, dt$. The inequality (1.8) provides a generalisation of a result obtained by Pachpatte in [18].

The main aim of this paper is to obtain some weighted inequalities of the Ostrowski type by providing various upper bounds for the deviation of $f(x)$, $x \in [a,b]$, from the integral mean

$$\frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt,$$

when f is absolutely continuous, of bounded variation or Lipschitzian on the interval $[a,b]$. Some particular cases of interest are also given.

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2. Ostrowski Type Inequalities

In order to state some Ostrowski type inequalities, we consider the Lebesgue norms

$$\|g\|_{[\alpha,\beta],\infty} := ess \sup_{t\in[\alpha,\beta]} |g(t)|$$

and

$$\|g\|_{[\alpha,\beta],\ell} := \left[\int_\alpha^\beta |g(t)|^\ell \, dt\right]^{\frac{1}{\ell}}, \quad \ell \in [1,\infty);$$

provided that the integral and the supremum are finite.

Theorem 2.1. *Let* $\varphi : [0,1] \to \mathbb{R}$ *be continuous on* $[0,1]$, *differentiable on* $(0,1)$ *with the property that* $\varphi(0) = 0$ *and* $\varphi(1) \neq 0$. *If* $w : [a,b] \to \mathbb{R}_+$ *is a probability density function, then for any* $f : [a,b] \to \mathbb{R}$ *an absolutely continuous function, we have*

$$(2.1)\quad \left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right|$$
$$\leq \int_a^x \left|\varphi\left(\int_a^t w(s)\,ds\right)\right| |f'(t)|\,dt + \int_x^b \left|\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right| |f'(t)|\,dt$$

for any $x \in [a,b]$.

If

$$H_1(x) := \int_a^x \left|\varphi\left(\int_a^t w(s)\,ds\right)\right| |f'(t)|\,dt$$

and

$$H_2(x) := \int_x^b \left|\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right| |f'(t)|\,dt,$$

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then

$$(2.2)\qquad H_1(x) \le \begin{cases} \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right)\right\|_{[a,x],\infty} \|f'\|_{[a,x],1}; & \\ \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right)\right\|_{[a,x],p} \|f'\|_{[a,x],q} & \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } f' \in L_q[a,x]; \\ \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right)\right\|_{[a,x],1} \|f'\|_{[a,x],\infty} & \text{if } f' \in L_\infty[a,x]; \end{cases}$$

and

$$(2.3)\qquad H_2(x) \le \begin{cases} \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right) - \varphi(1)\right\|_{[x,b],\infty} \|f'\|_{[x,b],1}; & \\ \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right) - \varphi(1)\right\|_{[x,b],r} \|f'\|_{[x,b],t} & \text{if } r>1, \frac{1}{r}+\frac{1}{t}=1 \\ & \text{and } f' \in L_t[x,b]; \\ \left\| \varphi\left(\int_a^{\cdot} w(s)\,ds\right) - \varphi(1)\right\|_{[x,b],1} \|f'\|_{[x,b],\infty} & \text{if } f' \in L_\infty[x,b] \end{cases}$$

for any $x \in [a,b]$.

Proof. Follows from the identity (1.1) on observing that

$$(2.4)\qquad \left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right|$$

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$$
\begin{aligned}
&= \left| \int_a^x \varphi\left(\int_a^t w(s)\,ds \right) f'(t)\,dt + \int_x^b \left[\varphi\left(\int_a^t w(s)\,ds \right) - \varphi(1) \right] f'(t)\,dt \right| \\
&\le \left| \int_a^x \varphi\left(\int_a^t w(s)\,ds \right) f'(t)\,dt \right| + \left| \int_x^b \left[\varphi\left(\int_a^t w(s)\,ds \right) - \varphi(1) \right] f'(t)\,dt \right| \\
&\le \int_a^x \left| \varphi\left(\int_a^t w(s)\,ds \right) \right| \left| f'(t) \right| dt + \int_x^b \left| \varphi\left(\int_a^t w(s)\,ds \right) - \varphi(1) \right| \left| f'(t) \right| dt
\end{aligned}
$$

for any $x \in [a,b]$, and the first part of (2.1) is proved.

The bounds from (2.2) and (2.3) follow by the Hölder inequality. □

Remark 1. It is obvious that, the above theorem provides 9 possible upper bounds for the absolute value of the deviation of $f(x)$ from the integral mean,

$$
\frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\,ds \right) f(t)\,dt
$$

although they are not stated explicitly.

The above result, which provides an Ostrowski type inequality for the absolutely continuous function f, can be extended to the larger class of functions of bounded variation as follows:

Theorem 2.2. *Let φ and w be as in Theorem 2.1. If w is continuous on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$, then:*

$$
(2.5) \qquad \left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\,ds \right) f(t)\,dt \right|
$$

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$$\leq \frac{1}{\varphi(1)} \left[\sup_{t\in[a,x]} \left| \varphi\left(\int_a^t w(s)\,ds\right)\right| \cdot \bigvee_a^x (f) \right.$$
$$\left. + \sup_{t\in[x,b]} \left| \varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1) \right| \cdot \bigvee_x^b (f) \right]$$
$$\leq \frac{1}{\varphi(1)} \cdot \max\left\{ \sup_{t\in[a,x]} \left| \varphi\left(\int_a^t w(s)\,ds\right)\right|, \right.$$
$$\left. \sup_{t\in[x,b]} \left| \varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1) \right| \right\} \cdot \bigvee_a^b (f),$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on* $[a,b]$.

Proof. We recall that, if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous on $[\alpha,\beta]$ and $v : [\alpha,\beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$\left| \int_\alpha^\beta p(t)\,dv(t) \right| \leq \sup_{t\in[\alpha,\beta]} |p(t)| \bigvee_\alpha^\beta (v). \tag{2.6}$$

Since the functions $\varphi\left(\int_a^\cdot w(s)\,ds\right)$ and $\varphi\left(\int_a^\cdot w(s)\,ds\right) - \varphi(1)$ are continuous on $[a,x]$ and $[x,b]$, respectively, the Riemann-Stieltjes integrals

$$\int_a^x \varphi\left(\int_a^t w(s)\,ds\right) df(t) \quad \text{and} \quad \int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t)$$

exist and

$$\left| \int_a^x \varphi\left(\int_a^t w(s)\,ds\right) df(t) \right| \leq \sup_{t\in[a,x]} \left| \varphi\left(\int_a^t w(s)\,ds\right)\right| \cdot \bigvee_a^x (f), \tag{2.7}$$

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while

$$\left|\int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t)\right| \leq \sup_{t\in[x,b]} \left|\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right| \cdot \bigvee_x^b (f). \tag{2.8}$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^x \varphi\left(\int_a^t w(s)\,ds\right) df(t) &= f(t)\,\varphi\left(\int_a^t w(s)\,ds\right)\Big|_a^x - \int_a^x f(t)\,d\left[\varphi\left(\int_a^t w(s)\,ds\right)\right] \\ &= f(x)\,\varphi\left(\int_a^x w(s)\,ds\right) - \int_a^x w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t) &= \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] f(t)\Big|_x^b \\ &\quad - \int_x^b f(t)\,d\left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] \\ &= -\left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] f(x)\end{aligned} \tag{2.10}$$

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$$-\int_x^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt.$$

If we add (2.9) and (2.10) we deduce the following identity of the Montgomery type for the Riemann-Stieltjes integral which is of interest in itself:

$$\begin{aligned} f(x) = {} & \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \\ & + \frac{1}{\varphi(1)} \int_a^x \varphi\left(\int_a^t w(s)\,ds\right) df(t) \\ & + \frac{1}{\varphi(1)} \int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t), \end{aligned} \tag{2.11}$$

for any $x \in [a,b]$.

Now, by (2.11) and (2.7) – (2.8) we obtain the estimate:

$$\begin{aligned} & \left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \\ & \le \frac{1}{\varphi(1)} \left| \int_a^x \varphi\left(\int_a^t w(s)\,ds\right) df(t) \right| + \frac{1}{\varphi(1)} \left| \int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t) \right| \\ & \le \frac{1}{\varphi(1)} \cdot \sup_{t\in[a,x]} \left| \varphi\left(\int_a^t w(s)\,ds\right) \right| \cdot \bigvee_a^x (f) \\ & \qquad + \frac{1}{\varphi(1)} \cdot \sup_{t\in[x,b]} \left| \varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1) \right| \cdot \bigvee_x^b (f), \qquad x \in [a,b] \end{aligned}$$

which provides the first inequality in (2.5).

The last part of (2.5) is obvious. □

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The following particular case is of interest for applications.

Corollary 2.3. *Assume that f,φ,w are as in Theorem 2.2. In addition, if φ is monotonic nondecreasing on $[0,1]$, then*

$$\left|f(x)-\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt\right| \tag{2.12}$$
$$\leq \frac{\varphi\left(\int_a^x w(s)\,ds\right)}{\varphi(1)}\cdot\bigvee_a^x(f)+\left[1-\frac{\varphi\left(\int_a^x w(s)\,ds\right)}{\varphi(1)}\right]\cdot\bigvee_x^b(f)$$
$$\leq\left[\frac{1}{2}+\left|\frac{\varphi\left(\int_a^x w(s)\,ds\right)}{\varphi(1)}-\frac{1}{2}\right|\right]\bigvee_a^b(f).$$

Proof. Follows by Theorem 2.2 on observing that, if φ is monotonic nondecreasing on $[a,b]$, then:

$$\sup_{t\in[a,x]}\left|\varphi\left(\int_a^t w(s)\,ds\right)\right|=\sup_{t\in[a,x]}\varphi\left(\int_a^t w(s)\,ds\right)=\varphi\left(\int_a^x w(s)\,ds\right)$$

and

$$\sup_{t\in[x,b]}\left|\varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1)\right|=\sup_{t\in[x,b]}\left[\varphi(1)-\varphi\left(\int_a^t w(s)\,ds\right)\right]$$
$$=\varphi(1)-\inf_{t\in[x,b]}\varphi\left(\int_a^t w(s)\,ds\right)$$
$$=\varphi(1)-\varphi\left(\int_a^x w(s)\,ds\right).$$

□

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Corollary 2.4. *With the assumptions of Theorem 2.2 and if* $K := \sup_{t\in(0,1)} |\varphi'(t)| < \infty$, *then we have the bounds:*

$$\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \tag{2.13}$$
$$\leq \frac{1}{\varphi(1)} \cdot K \left[\sup_{t\in[a,x]} \left| \int_a^t w(s)\,ds \right| \cdot \bigvee_a^x (f) + \sup_{t\in[x,b]} \left| \int_t^b w(s)\,ds \right| \cdot \bigvee_x^b (f) \right]$$
$$\leq \frac{K}{\varphi(1)} \max\left\{ \sup_{t\in[a,x]} \left| \int_a^t w(s)\,ds \right|, \sup_{t\in[x,b]} \left| \int_t^b w(s)\,ds \right| \right\} \bigvee_a^b (f).$$

Remark 2. If $w(s) \geq 0$ for $s \in [a,b]$, then from (2.13) we get

$$\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \tag{2.14}$$
$$\leq \frac{K}{\varphi(1)} \left[\int_a^x w(s)\,ds \cdot \bigvee_a^x (f) + \int_x^b w(s)\,ds \cdot \bigvee_x^b (f) \right]$$
$$\leq \frac{K}{\varphi(1)} \left[\frac{1}{2} \int_a^b w(s)\,ds + \frac{1}{2} \left| \int_a^x w(s)\,ds - \int_x^b w(s)\,ds \right| \right] \cdot \bigvee_a^b (f).$$

The following result, that provides an Ostrowski type inequality for $L-$Lipschitzian functions, can be stated as well.

Theorem 2.5. *Let* φ *and* w *be as in Theorem 2.1. If* w *is continuous on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *is an* L_1-*Lipschitzian function on* $[a,x]$ *and* L_2-*Lipschitzian on*

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$[x,b]$, *with* $x\in[a,b]$, *then*

$$\left|f(x)-\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt\right| \tag{2.15}$$
$$\leq \frac{1}{\varphi(1)}\left[L_1\cdot\int_a^x\left|\varphi\left(\int_a^t w(s)\,ds\right)\right|dt + L_2\cdot\int_x^b\left|\varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1)\right|dt\right]$$
$$\leq \max\{L_1,L_2\}\cdot\frac{1}{\varphi(1)}\left[\int_a^x\left|\varphi\left(\int_a^t w(s)\,ds\right)\right|dt + \int_x^b\left|\varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1)\right|dt\right].$$

Proof. We recall that, if $p:[\alpha,\beta]\to\mathbb{R}$ is $L-$Lipschitzian and v is Riemann integrable, then the Riemann-Stieltjes integral $\int_\alpha^\beta f(t)\,du(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right|\leq L\int_\alpha^\beta|p(t)|\,dt. \tag{2.16}$$

Now, if we apply the above property to the integrals

$$\int_a^x\varphi\left(\int_a^t w(s)\,ds\right)df(t)\quad\text{and}\quad\int_\alpha^b\left[\varphi\left(\int_a^t w(t)\,ds\right)-\varphi(1)\right]df(t),$$

then we can state that

$$\left|\int_a^x\varphi\left(\int_a^t w(s)\,ds\right)df(t)\right|\leq L_1\cdot\int_a^x\left|\varphi\left(\int_a^t w(s)\,ds\right)\right|dt \tag{2.17}$$

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and

$$\left|\int_x^b \left[\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right] df(t)\right| \le L_2 \cdot \int_x^b \left|\varphi\left(\int_a^t w(s)\,ds\right) - \varphi(1)\right| dt. \tag{2.18}$$

By making use of the identity (2.11), by (2.17) and (2.18) we deduce the first part of (2.15).

The last part is obvious. □

The following particular case is of interest as well.

Corollary 2.6. *With the assumptions of Theorem 2.5 and if* $K := \sup_{t\in(0,1)} |\varphi'(t)| < \infty$, *then*

$$\begin{aligned} &\left|f(x) - \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt\right| \\ &\le \frac{K}{\varphi(1)}\left[L_1 \cdot \int_a^x \left|\int_a^t w(s)\,ds\right| dt + L_2 \cdot \int_x^b \left|\int_t^b w(s)\,ds\right| dt\right] \\ &\le \frac{K}{\varphi(1)} \max\{L_1, L_2\} \left[\int_a^x \left|\int_a^t w(s)\,ds\right| dt + \int_x^b \left|\int_t^b w(s)\,ds\right| dt\right]. \end{aligned} \tag{2.19}$$

Remark 3. If $w : [a,b] \to \mathbb{R}$ is a nonnegative weight, then $\int_a^t w(s)\,ds, \int_t^b w(s)\,ds \ge 0$ for each $t \in [a,b]$ and since

$$\begin{aligned} \int_a^x \left(\int_a^t w(s)\,ds\right) dt &= \left(\int_a^t w(s)\,ds\right) \cdot t\Big|_a^x - \int_a^x w(t)\,dt \\ &= x\int_a^x w(t)\,dt - \int_a^x t w(t)\,dt = \int_a^x (x-t)\,w(t)\,dt \end{aligned}$$

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and

$$\int_x^b \left(\int_t^b w(s)\,ds\right) dt = t \cdot \left(\int_t^b w(s)\,ds\right)\Big|_x^b + \int_x^b w(t)\,dt$$
$$= -x\int_x^b w(t)\,dt + \int_x^b tw(t)\,dt = \int_x^b (t-x)\,w(t)\,dt,$$

then we get, from (2.19), the following result:

$$\left|f(x) - \frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt\right| \tag{2.20}$$
$$\leq \frac{K}{\varphi(1)}\left[L_1 \cdot \int_a^x (x-t)\,w(t)\,dt + L_2 \cdot \int_x^b (t-x)\,w(t)\,dt\right]$$
$$\leq \frac{K}{\varphi(1)}\max\{L_1, L_2\}\int_a^b |t-x|\,w(t)\,dt.$$

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3. Some Examples

The inequality (2.12) is a source of numerous particular inequalities that can be obtained by specifying the function $\varphi : [0,1] \to \mathbb{R}$ which is continuous, differentiable and monotonic nondecreasing with $\varphi(0) = 0$.

For instance, if we choose $\varphi(t) = t^{\alpha}$, $\alpha > 0$, then we get the inequality:

$$\left| f(x) - \alpha \int_a^b w(t) \left(\int_a^t w(s)\,ds \right)^{\alpha-1} f(t)\,dt \right| \tag{3.1}$$
$$\leq \left(\int_a^x w(s)\,ds \right)^{\alpha} \cdot \bigvee_a^x (f) + \left[1 - \left(\int_a^x w(s)\,ds \right)^{\alpha} \right] \cdot \bigvee_x^b (f)$$
$$\leq \left[\frac{1}{2} + \left| \left(\int_a^x w(s)\,ds \right)^{\alpha} - \frac{1}{2} \right| \right] \bigvee_a^b (f),$$

for any $x \in [a,b]$ provided that f is of bounded variation on $[a,b]$, $w(s) \geq 0$ for any $s \in [a,b]$ and the involved integrals exist.

Another simple example can be given by choosing $\varphi(t) = \ln(t+1)$. In this situation, we obtain the inequality:

$$\left| f(x) - \frac{1}{\ln 2} \int_a^b \left[\frac{w(t)}{\int_a^t w(s)\,ds + 1} \right] f(t)\,dt \right| \tag{3.2}$$
$$\leq \frac{\ln\left(\int_a^x w(s)\,ds + 1 \right)}{\ln 2} \cdot \bigvee_a^x (f) + \left[1 - \frac{\ln\left(\int_a^x w(s)\,ds + 1 \right)}{\ln 2} \right] \cdot \bigvee_x^b (f)$$
$$\leq \left[\frac{1}{2} + \left| \frac{\ln\left(\int_a^x w(s)\,ds + 1 \right)}{\ln 2} - \frac{1}{2} \right| \right] \cdot \bigvee_a^b (f),$$

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for any $x \in [a,b]$ provided that f is of bounded variation on $[a,b]$, $w(s) \geq 0$ for any $s \in [a,b]$ and the involved integrals exist.

Finally, by choosing the function $\varphi(t) = \exp(t) - 1$, we obtain, from the inequality (2.12), the following result as well:

$$\begin{aligned}
&\left| f(x) - \frac{1}{e-1} \int_a^b w(t) \exp\left(\int_a^t w(s)\,ds \right) f(t)\,dt \right| \\
&\leq \frac{\exp\left(\int_a^x w(s)\,ds\right) - 1}{e-1} \cdot \bigvee_a^x (f) + \frac{e - \exp\left(\int_a^x w(s)\,ds\right)}{e-1} \cdot \bigvee_x^b (f) \\
&\leq \left[\frac{1}{2} + \left| \frac{\exp\left(\int_a^x w(s)\,ds\right) - 1}{e-1} - \frac{1}{2} \right| \right] \cdot \bigvee_a^b (f),
\end{aligned}$$

for any $x \in [a,b]$, provided f is of bounded variation on $[a,b]$ and the involved integrals exist.

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