A NEW INEQUALITY FOR WEAKLY (K_1, K_2)-QUASIREGULAR MAPPINGS

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Received: 17 May, 2007

Accepted: 05 July, 2007

Communicated by: C. Bandle

2000 AMS Sub. Class.: 30C65, 35C60.

Key words: (K_1, K_2)-Quasiregular Mappings, regularity, Caccioppoli inequality.

Abstract: We obtain a new Caccioppoli inequality for weakly (K_1, K_2)-quasiregular mappings, which can be used to derive the self-improving regularity of (K_1, K_2)-Quasiregular Mappings.

Acknowledgements: Research supported by Doctoral Foundation of the Department of Education of Hebei Province (B2004103).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$, $n \geq 2$ and $0 \leq K_1, K_2 \leq \infty$ be two constants. Then a mapping $f \in W^{1,q}_{loc}(\Omega, \mathbf{R}^n)$, $(1 \leq q < \infty)$ is said to be weakly (K_1, K_2)-quasiregular, if $J(x, f) \geq 0$, a.e. Ω and

$$|Df(x)|^n \leq K_1 n^{n/2} J(x, f) + K_2, \text{ a.e. } x \in \Omega \tag{1.1}$$

where $|Df(x)| = \sup_{|h|=1} |Df(x)h|$ is the operator norm of the matrix $Df(x)$, the differential of f at the point x, and $J(x, f)$ is the Jacobian of f. If $q \geq n$, then f is called (K_1, K_2)-quasiregular. The word *weakly* in the definition means the Sobolev integrable exponent q of f may be smaller than the dimension n. In this case, $J(x, f)$ need not be locally integrable.

The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics such as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations (see [5] and the references therein).

Simon [7] established the Hölder continuity estimate when he studied the (K_1, K_2)-quasiconformal mappings between two surfaces of the Euclidean space $\mathbf{R}^3$. This estimate has important applications to elliptic partial differential equations with two variables. In [4], Gilbarg and Trudinger obtained an *a priori* $C^{1,\alpha}_{loc}$ estimate for quasi-linear elliptic equations with two variables by using the Hölder continuity method established in the studying of plane (K_1, K_2)-quasiregular mappings, and then established the existence theorem of the Dirichlet boundary value problem. Because of the importance of plane (K_1, K_2)-quasiregular mappings to the a priori estimates in nonlinear partial differential equation theory, Zheng and Fang [8] generalized (K_1, K_2)-quasiregular mappings from plane to space in 1998 by using the outer differential forms. Gao [2] generalized the result of [8] by obtaining the regularity result of weakly (K_1, K_2)-quasiregular mappings.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

A remarkable feature of (K_1, K_2)-quasiregular mappings is their self-improving regularity. In 1957 [1], Bojarski proved that for planar $(K_1, 0)$-quasiregular mappings, there exists an exponent $p(2, K) > 2$ such that $(K_1, 0)$-quasiregular mappings *a priori* in $W^{1,2}$ belong to $W^{1,p}$ for every $p < p(2, K)$. In 1973, Gehring [3] extended the result to n-dimensional $(K_1, 0)$-quasiconformal mappings (homeomorphic $(K_1, 0)$-quasiregular mappings) and proved the celebrated Gehring's Lemma. A bit later, Meyers and Elcrat [6] proved that Gehring's idea can be further exploited to treat quasiregular mappings and partial differential systems.

In this note, we give a new inequality for (K_1, K_2)-quasiregular mappings, from which one can derive self-improving regularity.

Theorem 1.1. *There exist two numbers $q(n, K) < n < p(n, K)$, such that for all s with $q(n, K) < s < p(n, K)$), every mapping $f \in W^{1,q}_{loc}(\Omega, \mathbf{R}^n)$ such that (1.1) holds belongs to $W^{1,s}_{loc}(\Omega, \mathbf{R}^n)$. Moreover, for each test function $\phi \in C_0^\infty(\Omega)$, we have the Caccioppoli-type inequality*

$$\|\phi Df\|_s \leq C_s(n, K_1, K_2)\|f \otimes \nabla\phi\|_s, \tag{1.2}$$

where $\otimes$ denotes the tensor product and $C(n, K_1, K_2)$ is a constant depending on n, K_1 and K_2.

Remark 1. By (1.2) and applying the classical Poincaré inequality, one infers that $|Df|^q$ satisfies a weak reverse Hölder's inequality. Then Gehring's lemma can be applied to verify the $L^{q+\delta}$ integrability of $|Df|$ with some $\delta = \delta(n, K) > 0$. The exponent will eventually exceed n by iterating the process, and the theorem is proved. The detailed argument is in [5, Theorem 17.3.1]. Therefore, we need only to prove inequality (1.2).

In order to prove Theorem 1.1, we need the following lemma [5, Theorem 7.8.2].

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issn: 1443-5756

Lemma 1.2. *Let a distribution* $f = (f^1, f^2, ..., f^n) \in D'(\mathbf{R}^n, \mathbf{R}^n)$ *have its differential* Df *in* $L^p(\mathbf{R}^n, \mathbf{R}^{n\times n})$, $1 \le p < \infty$. *Then*

$$\left| \int |Df(x)|^{p-n} J(x,f) dx \right| \le \lambda(n) \left| 1 - \frac{n}{p} \right| \int |Df(x)|^p dx.$$

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issn: 1443-5756

2. Proof of Theorem 1.1

Proof. We may assume that ϕ is non-negative as otherwise we could consider $|\phi|$ which has no effect on inequality (1.1). We can therefore write

$$|\phi Df|^p \le K_1 n^{n/2} |\phi Df|^{p-n} \det(\phi Df) + K_2 |\phi Df|^{p-n} \tag{2.1}$$

and introduce the auxiliary mapping

$$h = \phi f \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^n). \tag{2.2}$$

Since $Dh = \phi Df + f \otimes \nabla\phi$, inequality (2.1) can be expressed as

$$\begin{aligned} |Dh - f \otimes \nabla\phi|^p \le K_1 n^{n/2} |Dh - f \otimes \nabla\phi|^{p-n} \det(Dh - f \otimes \nabla\phi) \\ + K_2 |Dh - f \otimes \nabla\phi|^{p-n}. \end{aligned} \tag{2.3}$$

This gives us a non-homogeneous distortion inequality for h in $\mathbf{R}^n$:

$$|Dh|^p \le K_1 n^{n/2} |Dh|^{p-n} \det Dh + F + K_2 |Dh - f \otimes \nabla\phi|^{p-n}, \tag{2.4}$$

where

$$|F| \le C_p(n) K_1 n^{n/2} \big(|Dh| + |f \otimes \nabla\phi|\big)^{p-1} |f \otimes \nabla\phi|. \tag{2.5}$$

If we now apply Lemma 1.2, we obtain

$$\begin{aligned} \int_{\mathbf{R}^n} |Dh|^p \le \lambda K_1 n^{n/2} \left|1 - \frac{n}{p}\right| \int_{\mathbf{R}^n} |Dh|^p \\ + \int_{\mathbf{R}^n} |F| + K_2 \int_{\mathbf{R}^n} |Dh - f \otimes \nabla\phi|^{p-n}. \end{aligned} \tag{2.6}$$

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Hence

$$(2.7)\qquad \int_{\mathbf{R}^n}|Dh|^p \le \frac{C_p(n)K_1 n^{n/2}}{1-\lambda K_1 n^{n/2}\left|1-\frac{n}{p}\right|}\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^{p-1}|f\otimes\nabla\phi| + \frac{K_2}{1-\lambda K_1 n^{n/2}\left|1-\frac{n}{p}\right|}\int_{\mathbf{R}^n}|Dh-f\otimes\nabla\phi|^{p-n}.$$

We add $\int|f\otimes\nabla\phi|^p$ to both sides of this equation, and after a little manipulation we have

$$(2.8)\qquad \begin{aligned}&\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^p\\ &\le C_p(n,K_1)\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^{p-1}|f\otimes\nabla\phi|\\ &\qquad + C_p(n,K_1,K_2)\int_{\mathbf{R}^n}|Dh-f\otimes\nabla\phi|^{p-n}\\ &\le C_p(n,K_1)\left[\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^p\right]^{\frac{p-1}{p}}\left[\int_{\mathbf{R}^n}|f\otimes\nabla\phi|^p\right]^{\frac{1}{p}}\\ &\qquad + C_p(n,K_1,K_2)\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^p.\end{aligned}$$

Hence

$$(2.9)\qquad \left[\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^p\right]^{\frac{1}{p}} \le C_p(n,K_1)\left[\int_{\mathbf{R}^n}|f\otimes\nabla\phi|^p\right]^{\frac{1}{p}} + C_p(n,K_1,K_2)\left[\int_{\mathbf{R}^n}\left(|Dh|+|f\otimes\nabla\phi|\right)^p\right]^{\frac{1}{p}},$$

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issn: 1443-5756

that is

$$\Big\| |Dh| + |f \otimes \nabla\phi| \Big\|_p \le C_p(n, K_1)\|f \otimes \nabla\phi\|_p + C_p(n, K_1, K_2)\Big\| |Dh| + |f \otimes \nabla\phi| \Big\|_p. \tag{2.10}$$

Then, in view of the simple fact that $|\phi Df| \le |Dh| + |f \otimes \nabla\phi|$, we obtain the Caccioppoli-type estimate

$$\|\phi Df\|_p \le C_p(n, K_1, K_2)\|f \otimes \nabla\phi\|_p.$$

Of course, now we observe that this inequality holds with p replaced by s for any s in the range $q(n, K) \le s \le p(n, K)$, provided we know *a priori* that $f \in W^{1,s}_{loc}(\Omega, \mathbf{R}^n)$. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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vol. 8, iss. 3, art. 91, 2007

journal of inequalities in pure and applied mathematics

issn: 1443-5756