

A NEW INEQUALITY FOR WEAKLY (K_1, K_2) -QUASIREGULAR MAPPINGS

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Received: 17 May, 2007

Accepted: 05 July, 2007

Communicated by: C. Bandle

2000 AMS Sub. Class.: 30C65, 35C60.

Key words: (K_1, K_2) -Quasiregular Mappings, regularity, Caccioppoli inequality.

Abstract: We obtain a new Caccioppoli inequality for weakly (K_1, K_2) -quasiregular mappings, which can be used to derive the self-improving regularity of (K_1, K_2) -Quasiregular Mappings.

Acknowledgements: Research supported by Doctoral Foundation of the Department of Education of Hebei Province (B2004103).



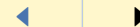
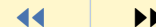
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Quasiregular Mappings

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1. Introduction

Let Ω be a bounded domain of \mathbf{R}^n , $n \geq 2$ and $0 \leq K_1, K_2 \leq \infty$ be two constants. Then a mapping $f \in W_{loc}^{1,q}(\Omega, \mathbf{R}^n)$, ($1 \leq q < \infty$) is said to be weakly (K_1, K_2) -quasiregular, if $J(x, f) \geq 0$, a.e. Ω and

$$(1.1) \quad |Df(x)|^n \leq K_1 n^{n/2} J(x, f) + K_2, \text{ a.e. } x \in \Omega$$

where $|Df(x)| = \sup_{|h|=1} |Df(x)h|$ is the operator norm of the matrix $Df(x)$, the differential of f at the point x , and $J(x, f)$ is the Jacobian of f . If $q \geq n$, then f is called (K_1, K_2) -quasiregular. The word *weakly* in the definition means the Sobolev integrable exponent q of f may be smaller than the dimension n . In this case, $J(x, f)$ need not be locally integrable.

The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics such as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations (see [5] and the references therein).

Simon [7] established the Hölder continuity estimate when he studied the (K_1, K_2) -quasiconformal mappings between two surfaces of the Euclidean space \mathbf{R}^3 . This estimate has important applications to elliptic partial differential equations with two variables. In [4], Gilbarg and Trudinger obtained an *a priori* $C_{loc}^{1,\alpha}$ estimate for quasilinear elliptic equations with two variables by using the Hölder continuity method established in the studying of plane (K_1, K_2) -quasiregular mappings, and then established the existence theorem of the Dirichlet boundary value problem. Because of the importance of plane (K_1, K_2) -quasiregular mappings to the a priori estimates in nonlinear partial differential equation theory, Zheng and Fang [8] generalized (K_1, K_2) -quasiregular mappings from plane to space in 1998 by using the outer differential forms. Gao [2] generalized the result of [8] by obtaining the regularity result of weakly (K_1, K_2) -quasiregular mappings.



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A remarkable feature of (K_1, K_2) -quasiregular mappings is their self-improving regularity. In 1957 [1], Bojarski proved that for planar $(K_1, 0)$ -quasiregular mappings, there exists an exponent $p(2, K) > 2$ such that $(K_1, 0)$ -quasiregular mappings *a priori* in $W^{1,2}$ belong to $W^{1,p}$ for every $p < p(2, K)$. In 1973, Gehring [3] extended the result to n -dimensional $(K_1, 0)$ -quasiconformal mappings (homeomorphic $(K_1, 0)$ -quasiregular mappings) and proved the celebrated Gehring's Lemma. A bit later, Meyers and Elcrat [6] proved that Gehring's idea can be further exploited to treat quasiregular mappings and partial differential systems.

In this note, we give a new inequality for (K_1, K_2) -quasiregular mappings, from which one can derive self-improving regularity.

Theorem 1.1. *There exist two numbers $q(n, K) < n < p(n, K)$, such that for all s with $q(n, K) < s < p(n, K)$, every mapping $f \in W_{loc}^{1,q}(\Omega, \mathbf{R}^n)$ such that (1.1) holds belongs to $W_{loc}^{1,s}(\Omega, \mathbf{R}^n)$. Moreover, for each test function $\phi \in C_0^\infty(\Omega)$, we have the Caccioppoli-type inequality*

$$(1.2) \quad \|\phi Df\|_s \leq C_s(n, K_1, K_2) \|f \otimes \nabla \phi\|_s,$$

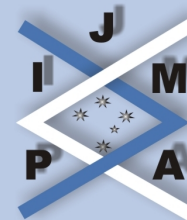
where \otimes denotes the tensor product and $C(n, K_1, K_2)$ is a constant depending on n, K_1 and K_2 .

Remark 1. By (1.2) and applying the classical Poincaré inequality, one infers that $|Df|^q$ satisfies a weak reverse Hölder's inequality. Then Gehring's lemma can be applied to verify the $L^{q+\delta}$ integrability of $|Df|$ with some $\delta = \delta(n, K) > 0$. The exponent will eventually exceed n by iterating the process, and the theorem is proved. The detailed argument is in [5, Theorem 17.3.1]. Therefore, we need only to prove inequality (1.2).

In order to prove Theorem 1.1, we need the following lemma [5, Theorem 7.8.2].

Lemma 1.2. *Let a distribution $f = (f^1, f^2, \dots, f^n) \in D'(\mathbf{R}^n, \mathbf{R}^n)$ have its differential Df in $L^p(\mathbf{R}^n, \mathbf{R}^{n \times n})$, $1 \leq p < \infty$. Then*

$$\left| \int |Df(x)|^{p-n} J(x, f) dx \right| \leq \lambda(n) \left| 1 - \frac{n}{p} \right| \int |Df(x)|^p dx.$$



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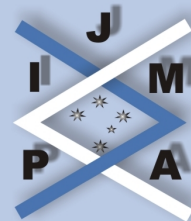
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2. Proof of Theorem 1.1

Proof. We may assume that ϕ is non-negative as otherwise we could consider $|\phi|$ which has no effect on inequality (1.1). We can therefore write

$$(2.1) \quad |\phi Df|^p \leq K_1 n^{n/2} |\phi Df|^{p-n} \det(\phi Df) + K_2 |\phi Df|^{p-n}$$

and introduce the auxiliary mapping

$$(2.2) \quad h = \phi f \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^n).$$

Since $Dh = \phi Df + f \otimes \nabla \phi$, inequality (2.1) can be expressed as

$$(2.3) \quad |Dh - f \otimes \nabla \phi|^p \leq K_1 n^{n/2} |Dh - f \otimes \nabla \phi|^{p-n} \det(Dh - f \otimes \nabla \phi) + K_2 |Dh - f \otimes \nabla \phi|^{p-n}.$$

This gives us a non-homogeneous distortion inequality for h in \mathbf{R}^n :

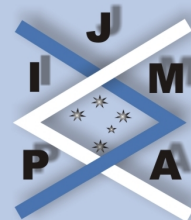
$$(2.4) \quad |Dh|^p \leq K_1 n^{n/2} |Dh|^{p-n} \det Dh + F + K_2 |Dh - f \otimes \nabla \phi|^{p-n},$$

where

$$(2.5) \quad |F| \leq C_p(n) K_1 n^{n/2} (|Dh| + |f \otimes \nabla \phi|)^{p-1} |f \otimes \nabla \phi|.$$

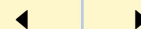
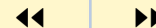
If we now apply Lemma 1.2, we obtain

$$(2.6) \quad \int_{\mathbf{R}^n} |Dh|^p \leq \lambda K_1 n^{n/2} \left| 1 - \frac{n}{p} \right| \int_{\mathbf{R}^n} |Dh|^p + \int_{\mathbf{R}^n} |F| + K_2 \int_{\mathbf{R}^n} |Dh - f \otimes \nabla \phi|^{p-n}.$$



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Hence

$$(2.7) \quad \int_{\mathbf{R}^n} |Dh|^p \leq \frac{C_p(n)K_1n^{n/2}}{1 - \lambda K_1n^{n/2} \left|1 - \frac{n}{p}\right|} \int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^{p-1} |f \otimes \nabla\phi| \\ + \frac{K_2}{1 - \lambda K_1n^{n/2} \left|1 - \frac{n}{p}\right|} \int_{\mathbf{R}^n} |Dh - f \otimes \nabla\phi|^{p-n}.$$

We add $\int |f \otimes \nabla\phi|^p$ to both sides of this equation, and after a little manipulation we have

$$(2.8) \quad \int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^p \\ \leq C_p(n, K_1) \int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^{p-1} |f \otimes \nabla\phi| \\ + C_p(n, K_1, K_2) \int_{\mathbf{R}^n} |Dh - f \otimes \nabla\phi|^{p-n} \\ \leq C_p(n, K_1) \left[\int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^p \right]^{\frac{p-1}{p}} \left[\int_{\mathbf{R}^n} |f \otimes \nabla\phi|^p \right]^{\frac{1}{p}} \\ + C_p(n, K_1, K_2) \int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^p.$$

Hence

$$(2.9) \quad \left[\int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^p \right]^{\frac{1}{p}} \leq C_p(n, K_1) \left[\int_{\mathbf{R}^n} |f \otimes \nabla\phi|^p \right]^{\frac{1}{p}} \\ + C_p(n, K_1, K_2) \left[\int_{\mathbf{R}^n} (|Dh| + |f \otimes \nabla\phi|)^p \right]^{\frac{1}{p}},$$

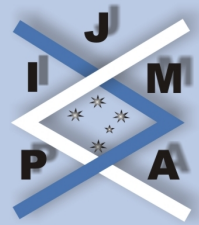
that is

$$(2.10) \quad \begin{aligned} & \| |Dh| + |f \otimes \nabla\phi| \|_p \\ & \leq C_p(n, K_1) \|f \otimes \nabla\phi\|_p + C_p(n, K_1, K_2) \| |Dh| + |f \otimes \nabla\phi| \|_p. \end{aligned}$$

Then, in view of the simple fact that $|\phi Df| \leq |Dh| + |f \otimes \nabla\phi|$, we obtain the Caccioppoli-type estimate

$$\|\phi Df\|_p \leq C_p(n, K_1, K_2) \|f \otimes \nabla\phi\|_p.$$

Of course, now we observe that this inequality holds with p replaced by s for any s in the range $q(n, K) \leq s \leq p(n, K)$, provided we know *a priori* that $f \in W_{loc}^{1,s}(\Omega, \mathbf{R}^n)$. \square



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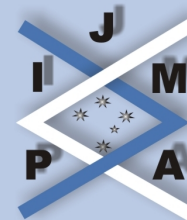
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