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AN INEQUALITY BETWEEN COMPOSITIONS OF WEIGHTED ARITHMETIC AND GEOMETRIC MEANS

FINBARR HOLLAND

Mathematics Department
University College
Cork, Ireland

EMail: f.holland@ucc.ie

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Abstract

Let \mathbb{P} denote the collection of positive sequences defined on \mathbb{N} . Fix $w \in \mathbb{P}$. Let s, t , respectively, be the sequences of partial sums of the infinite series $\sum w_k$ and $\sum s_k$, respectively. Given $x \in \mathbb{P}$, define the sequences $A(x)$ and $G(x)$ of weighted arithmetic and geometric means of x by

$$A_n(x) = \sum_{k=1}^n \frac{w_k}{s_n} x_k, \quad G_n(x) = \prod_{k=1}^n x_k^{w_k/s_n}, \quad n = 1, 2, \dots$$

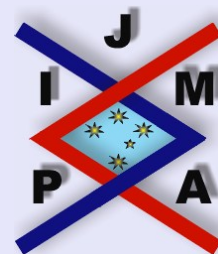
Under the assumption that $\log t$ is concave, it is proved that $A(G(x)) \leq G(A(x))$ for all $x \in \mathbb{P}$, with equality if and only if x is a constant sequence.

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1. Introduction

In [13], Kedlaya proved the following theorem.

Theorem 1.1. *Let $x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_n$ be positive real numbers, and define $s_i = w_1 + w_2 + \dots + w_i, i = 1, 2, \dots, n$. Assume that*

$$(1.1) \quad \frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \dots \geq \frac{w_n}{s_n}.$$

Then

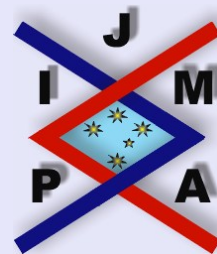
$$(1.2) \quad \prod_{i=1}^n \left(\sum_{j=1}^i \frac{w_j}{s_i} x_j \right)^{w_i/s_n} \geq \sum_{j=1}^n \frac{w_j}{s_n} \prod_{i=1}^j x_i^{w_i/s_j},$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Choosing w to be a constant sequence, we recover the inequality

$$(1.3) \quad \sqrt[n]{\prod_{i=1}^n \left(\frac{1}{i} \sum_{j=1}^i x_j \right)} \geq \frac{1}{n} \sum_{j=1}^n \sqrt[j]{\prod_{i=1}^j x_i},$$

which Kedlaya [12] had previously established, thereby confirming a conjecture of the author [9]. The strict inequality prevails in (1.3) unless $x_1 = x_2 = \dots = x_n$. Evidently, inequality (1.3) is a sharp refinement of Carleman's well-known one [4, 7]. (Indeed, as a tribute to Carleman, the author was led to formulate (1.3) in an attempt to design a suitable problem for the IMO when it was held



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in Sweden in 1991. However, unbeknownst to him at the time, two stronger versions of it had already been stated, without proof, by Nanjundiah [17].)

In passing, we note that (1.3) is also a simple consequence of more general results found by Bennett [2, 3], and Mond and Pečarić [16].

Also in [13], Kedlaya deduced a weighted version of Carleman's inequality from Theorem 1.1, viz.,

Theorem 1.2. *Let w_1, w_2, \dots be a sequence of positive real numbers, and define $s_i = w_1 + w_2 + \dots + w_i$, for $i = 1, 2, \dots$. Assume that*

$$(1.4) \quad \frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \dots .$$

Then, for any sequence a_1, a_2, \dots of positive real numbers with $\sum_k w_k a_k < \infty$,

$$\sum_{k=1}^{\infty} w_k a_1^{w_1/s_k} \dots a_k^{w_k/s_k} < e \sum_{k=1}^{\infty} w_k a_k .$$

Carleman's classical inequality is obtained from this by setting $w_i = 1$, $i = 1, 2, \dots$. This beautiful result has attracted the attention of many authors, and has been proved in a variety of ways. It has also been extended in different directions by a host of people. Anyone interested in knowing the history of Carleman's inequality, and such matters, is urged to consult [11], which has an extensive bibliography. In addition, the fascinating monograph by Bennett [1] contains some very interesting developments of it, and mentions, *inter alia*, the significant extensions of it made by Cochran and Lee [5], Heinig [8] and Love [14, 15]. Readers interested in its continuous analogues should also read [18].



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Kedlaya expressed a doubt that the monotonicity condition (1.4) was needed in Theorem 1.2. His suspicions were well-founded, for, already in 1925, Hardy [6, 7], following a suggestion made to him by Pólya, proved this statement without any extra hypothesis on the weights. In fact, in the presence of condition (1.4), a much stronger conclusion can be drawn, as the author has recently discovered [10]. This begs the question: does Theorem 1.1 also hold under less stringent conditions on the weights than (1.1)? It is trivially true when $n = 1$, and a convexity argument shows it also holds without any restriction on the weights when $n = 2$. However, as Kedlaya himself pointed out, the result is false in general. As he mentions, a necessary condition for the truth of Theorem 1.1 is that

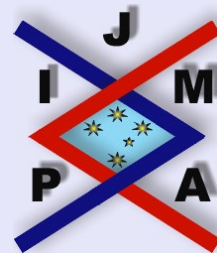
$$\left(\frac{w_n}{s_n}\right)^{s_{n-1}} \leq \left(\frac{w_1}{s_1}\right)^{w_1} \left(\frac{w_2}{s_2}\right)^{w_2} \cdots \left(\frac{w_{n-1}}{s_{n-1}}\right)^{w_{n-1}}.$$

On the other hand, examples show that the sufficient assumption (1.1) is not necessary. For instance, with $n = 3, w_1 = 2, w_2 = 1, w_3 = 3$, then $w_2/s_2 < w_3/s_3$, so that condition (1.1) fails, yet

$$\frac{2a + \sqrt[3]{a^2b} + 3\sqrt[6]{a^2bc^3}}{6} \leq \sqrt[6]{a^2 \left(\frac{2a+b}{3}\right) \left(\frac{2a+b+3c}{6}\right)^3},$$

for all $a, b, c > 0$, with equality if and only if $a = b = c$. (This is a simple consequence of the fact that, if

$$F(x, y) = \frac{(2 + x + 3\sqrt{xy})^6}{(2 + x^3)(2 + x^3 + 3y^2)^3},$$



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then

$$\begin{aligned} \max_{x \geq 0} \max_{y \geq 0} F(x, y) &= \max_{x \geq 0} \left[\frac{1}{2 + x^3} \left(\max_{y \geq 0} \frac{(2 + x + 3\sqrt{xy})^2}{2 + x^3 + 3y^2} \right)^3 \right] \\ &= \max_{x \geq 0} \frac{(4 + 10x + x^2 + 3x^4)^3}{(2 + x^3)^4} \\ &= 72, \end{aligned}$$

which can be verified in a routine manner, even by non-calculus arguments. Alternatively, it can be inferred as a special case of Theorem 2.1 which follows. Moreover, there is equality if and only if $x = y = 1$.)

As an examination of his proof of Theorem 1.1 reveals, Kedlaya actually proved something stronger than (1.2) under the hypothesis (1.1), namely, denoting by L_n, R_n the left-hand and right-hand sides of (1.2), then

$$(1.5) \quad \left(\frac{L_1}{R_1} \right)^{s_1} \leq \left(\frac{L_2}{R_2} \right)^{s_2} \leq \dots \leq \left(\frac{L_n}{R_n} \right)^{s_n}.$$

However, this statement is false in general, and, in particular, is not implied by (1.2). To see this, note that, with $n = 3$, and the same choice of weights $w_1 = 2, w_2 = 1, w_3 = 3$ as before, so that (1.2) holds, the claim that $(L_3/R_3)^{s_3} \geq (L_2/R_2)^{s_2}$ is equivalent to the statement that

$$2(2a + b + 3c)(2a + \sqrt[3]{a^2b}) \geq (2a + \sqrt[3]{a^2b} + 3\sqrt[6]{a^2bc^3})^2, \quad \forall a, b, c > 0.$$

However, this is not true generally, as may be seen by taking $a = 1, b = 64, c = 121$. So, Kedlaya proved a stronger statement with the hypothesis that the sequence s_i/w_i is increasing. By adopting a different proof-strategy, we show here that (1.2) holds under a weaker hypothesis than this.



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2. The Main Result

The purpose of this note is to present the following result which strengthens Theorem 1.1.

Theorem 2.1. *Let $x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_n$ be positive real numbers. Define $s_i = w_1 + w_2 + \dots + w_i$, $i = 1, 2, \dots, n$. Assume that*

$$(2.1) \quad \frac{s_k^2}{w_{k+1}} \geq \sum_{j=1}^{k-1} s_j, \quad k = 2, 3, \dots, n-1.$$

Then

$$\prod_{i=1}^n \left(\sum_{j=1}^i \frac{w_j}{s_i} x_j \right)^{w_i/s_n} \geq \sum_{j=1}^n \frac{w_j}{s_n} \prod_{i=1}^j x_i^{w_i/s_j}.$$

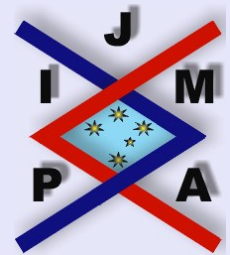
Equality holds if and only if $x_1 = x_2 = \dots = x_n$.

Remark 1. *In terms of the sequence $t_i = s_1 + s_2 + \dots + s_i$, $i = 1, 2, \dots, n$, it is not difficult to see that (2.1) is equivalent to the statement*

$$t_i^2 \geq t_{i-1}t_{i+1}, \quad i = 2, 3, \dots, n-1,$$

i.e., that $\log t_i$ is concave, whereas (1.1) is equivalent to the assertion that $\log s_i$ is concave. But we make no use of this alternative description of (2.1).

Before turning to the proof of Theorem 2.1 we show that (2.1) is implied by (1.1).



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Lemma 2.2. Let w_1, w_2, \dots be a sequence of positive numbers, and define the sequence s_1, s_2, \dots by

$$s_i = w_1 + w_2 + \dots + w_i, \quad i = 1, 2, \dots$$

Suppose

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \dots \geq \frac{w_n}{s_n} \geq \dots$$

Then

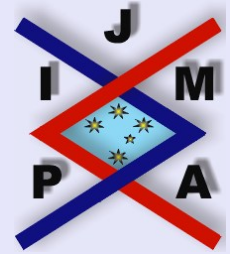
$$s_k^2 - w_{k+1} \sum_{j=1}^{k-1} s_j > 0, \quad k = 2, 3, \dots$$

Proof. The proof is by induction. To begin with, since $w_2 s_2 - w_3 s_1 = w_2 s_3 - w_3 s_2 \geq 0$, we have that

$$s_2^2 - w_3 s_1 = w_1 s_2 + w_2 s_2 - w_3 s_1 \geq w_1 s_2 > 0.$$

So, suppose the claimed result holds for some $m \geq 2$. Then, noting that, for $i \geq 2$, $w_i s_i - w_{i+1} s_{i-1} = w_i s_{i+1} - w_{i+1} s_i \geq 0$, we see that

$$\begin{aligned} s_{m+1}^2 - w_{m+2} \sum_{j=1}^m s_j &\geq \frac{w_{m+2}}{w_{m+1}} s_{m+1} s_m - w_{m+2} \sum_{j=1}^m s_j \\ &= \frac{w_{m+2}}{w_{m+1}} \left(s_{m+1} s_m - w_{m+1} \sum_{j=1}^m s_j \right) \\ &= \frac{w_{m+2}}{w_{m+1}} \left(s_m^2 + w_{m+1} s_m - w_{m+1} \sum_{j=1}^m s_j \right) \end{aligned}$$



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$$= \frac{w_{m+2}}{w_{m+1}} \left(s_m^2 - w_{m+1} \sum_{j=1}^{m-1} s_j \right) > 0,$$

by the induction assumption. The result follows. \square

We prove Theorem 2.1 by induction, and, to make productive use of the induction hypothesis, we need the following elementary result.

Lemma 2.3. *Let $A, B > 0$. Let $p > 1, q = p/(p - 1)$. Then, for all $s \geq 0$,*

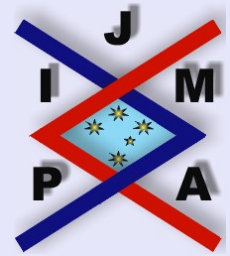
$$(A + Bs)^p \leq (A^q + B^q)^{p-1} (1 + s^p),$$

with equality if and only if $s = (B/A)^{q-1}$.

Proof. The inequality is trivial if $s = 0$. Suppose $s > 0$. Exploiting the strict convexity of $t \rightarrow t^q$, it is clear that

$$\begin{aligned} \left(\frac{A + Bs}{1 + s^p} \right)^q &= \left(\frac{A + (Bs^{1-p})s^p}{1 + s^p} \right)^q \\ &\leq \frac{A^q + (Bs^{1-p})^q s^p}{1 + s^p} \\ &= \frac{A^q + B^q}{1 + s^p}, \end{aligned}$$

with equality if and only if $A = Bs^{1-p}$. The stated result follows quickly from this. \square



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Corollary 2.4. Let $p > 1$, $q = p/(p - 1)$. Let $A, B, C, D > 0$. Then, for all $t \geq 0$,

$$\frac{(A + Bt)^p}{C + Dt^p} \leq \frac{1}{CD} (A^q D^{q-1} + B^q C^{q-1})^{p-1},$$

with equality if and only if $t = (BC/AD)^{q-1}$.

We are now ready to deal with the proof of Theorem 2.1.

For convenience, define the sequences of weighted averages A_k, G_k of x_1, x_2, \dots, x_n by

$$A_k = \sum_{i=1}^k \frac{w_i}{s_k} x_i, \quad G_k = \prod_{i=1}^k x_i^{w_i/s_k}, \quad k = 1, 2, \dots, n.$$

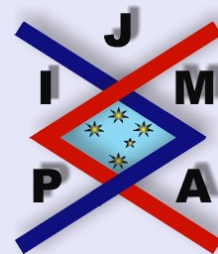
We are required to prove that

$$\sum_{i=1}^n \frac{w_i}{s_n} G_i \leq \prod_{i=1}^n A_i^{w_i/s_n},$$

holds under condition (2.1), with equality if and only if

$$x_1 = x_2 = \dots = x_n.$$

Proof. We prove this by induction. The result clearly holds for $n = 1$. Moreover, as we mentioned in the introduction, a simple convexity argument establishes that it also holds when $n = 2$. We continue, therefore, with the assumption that $n \geq 3$. Suppose the result holds for some positive integer m , with



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$1 \leq m \leq n - 1$, so that, with

$$X = \prod_{i=1}^m A_i^{w_i/s_m},$$

then

$$\begin{aligned} \sum_{i=1}^{m+1} \frac{w_i}{s_{m+1}} G_i &= \frac{s_m \sum_{i=1}^m \frac{w_i}{s_m} G_i + w_{m+1} G_{m+1}}{s_{m+1}} \\ &\leq \frac{s_m X + w_{m+1} G_{m+1}}{s_{m+1}} \\ &= (1 - \alpha)X + \alpha Y x_{m+1}^\alpha, \end{aligned}$$

where $\alpha = w_{m+1}/s_{m+1}$ and

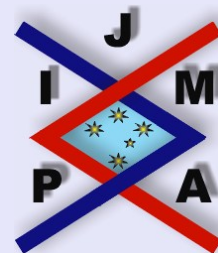
$$Y = \prod_{i=1}^m x_i^{w_i/s_{m+1}} = G_m^{s_m/s_{m+1}} = G_m^{1-\alpha}.$$

In addition,

$$A_{m+1} = \frac{s_m A_m + w_{m+1} x_{m+1}}{s_{m+1}} = (1 - \alpha)A_m + \alpha x_{m+1}.$$

We claim now that

$$\begin{aligned} (1 - \alpha)X + \alpha Y x_{m+1}^\alpha &\leq X^{s_m/s_{m+1}} A_{m+1}^{w_{m+1}/s_{m+1}} \\ &= X^{1-\alpha} ((1 - \alpha)A_m + \alpha x_{m+1})^\alpha, \end{aligned}$$



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i.e.,

$$\frac{((1 - \alpha)X + \alpha Y x_{m+1}^\alpha)^{1/\alpha}}{(1 - \alpha)A_m + \alpha x_{m+1}} \leq X^{(1-\alpha)/\alpha}.$$

By Corollary 2.4, with $p = 1/\alpha$, $A = (1 - \alpha)X$, $B = \alpha Y$, $C = (1 - \alpha)A_m$, $D = \alpha$, $q = 1/(1 - \alpha)$, the left-hand side does not exceed

$$\frac{\left((1 - \alpha)X^{1/(1-\alpha)} + \alpha Y^{1/(1-\alpha)} A_m^{\alpha/(1-\alpha)} \right)^{(1-\alpha)/\alpha}}{A_m},$$

with equality if and only if

$$x_{m+1} = \left(\frac{Y A_m}{X} \right)^{1/(1-\alpha)}.$$

Thus, to finish the proof, we must establish that

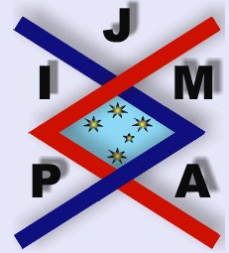
$$(1 - \alpha)X^{1/(1-\alpha)} + \alpha Y^{1/(1-\alpha)} A_m^{\alpha/(1-\alpha)} \leq X A_m^{\alpha/(1-\alpha)},$$

i.e., that

$$s_m \left(\frac{X}{A_m} \right)^{\alpha/(1-\alpha)} + w_{m+1} \frac{Y^{1/(1-\alpha)}}{X} \leq s_{m+1}.$$

In other words,

$$(2.2) \quad s_m \left(\frac{\prod_{i=1}^m A_i^{w_i/s_m}}{A_m} \right)^{w_{m+1}/s_m} + w_{m+1} \prod_{i=1}^m \left(\frac{x_i}{A_i} \right)^{w_i/s_m} \leq s_{m+1},$$



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with the additional assertion that there is equality if and only if $x_1 = x_2 = \dots = x_m$. This inequality is of independent interest, and can be considered for its own sake. To prove it, consider the second term on the left-hand side of (2.2). This is equal to

$$\frac{w_{m+1}G_m}{X} = w_{m+1} \sqrt[s_m]{\prod_{i=1}^m \left(\frac{x_i}{A_i}\right)^{w_i}},$$

whence, by the convexity of the exponential function, bearing in mind that $s_m = \sum_{i=1}^m w_i$, we see that this does not exceed

$$\frac{w_{m+1}}{s_m} \sum_{i=1}^m \frac{w_i x_i}{A_i}.$$

Moreover, there is equality if and only if

$$1 = \frac{x_1}{A_1} = \frac{x_i}{A_i}, \quad i = 1, 2, \dots, m,$$

i.e., $x_1 = x_2 = \dots = x_m$.

Now we focus on the first term. To begin with, observe that

$$\begin{aligned} \frac{X}{A_m} &= \sqrt[s_m]{\frac{\prod_{i=1}^m A_i^{w_i}}{A_m^{s_m}}} \\ &= \sqrt[s_m]{\frac{\prod_{i=1}^{m-1} A_i^{w_i}}{A_m^{s_m-1}}} = \sqrt[s_m]{\prod_{i=1}^{m-1} \left(\frac{A_i}{A_{i+1}}\right)^{s_i}}. \end{aligned}$$



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Hence, once more by the convexity of the exponential function,

$$\begin{aligned} s_m \left(\frac{X}{A_m} \right)^{\alpha/(1-\alpha)} &= s_m \left(1^{c_m} \prod_{i=1}^{m-1} \left(\frac{A_i}{A_{i+1}} \right)^{s_i} \right)^{w_{m+1}/s_m^2} \\ &\leq \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=1}^{m-1} \frac{s_i A_i}{A_{i+1}} \right) \\ &= \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=2}^m \frac{s_{i-1} A_{i-1}}{A_i} \right), \end{aligned}$$

where

$$c_m = \frac{s_m^2}{w_{m+1}} - \sum_{i=1}^{m-1} s_i \geq 0,$$

by hypothesis. Equality holds here if and only if

$$1 = \frac{A_i}{A_{i+1}}, \quad i = 1, 2, \dots, m-1,$$

i.e.,

$$s_i \sum_{j=1}^{i+1} w_j x_j = s_{i+1} \sum_{j=1}^i w_j x_j, \quad i = 1, 2, \dots, m-1,$$

equivalently, if and only if $x_m = \dots = x_2 = x_1$.



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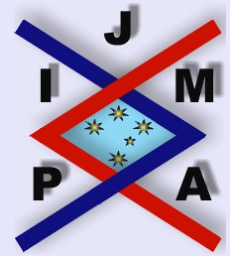
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Combining our estimates we see that

$$\begin{aligned}
 s_m \left(\frac{X}{A_m} \right)^{\alpha/(1-\alpha)} + w_{m+1} \frac{Y^{1/(1-\alpha)}}{X} &\leq \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=2}^m \frac{s_{i-1}A_{i-1}}{A_i} + \sum_{i=1}^m \frac{w_i x_i}{A_i} \right) \\
 &= \frac{w_{m+1}}{s_m} \left(c_m + w_1 + \sum_{i=2}^m \frac{s_{i-1}A_{i-1} + w_i x_i}{A_i} \right) \\
 &= \frac{w_{m+1}}{s_m} \left(c_m + w_1 + \sum_{i=2}^m \frac{s_i A_i}{A_i} \right) \\
 &= \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=1}^{m-1} s_i + s_m \right) \\
 &= \frac{w_{m+1}}{s_m} \left(\frac{s_m^2}{w_{m+1}} + s_m \right) \\
 &= s_{m+1}.
 \end{aligned}$$

Thus (2.2) holds. Moreover, equality holds in (2.2) if and only if $x_1 = x_2 = \dots = x_m$. Of course, (2.2) implies the inequality in Theorem 2.1, by induction. It therefore only remains to discuss the case of equality in this. But, if $x_1 = x_2 = \dots = x_m$, then $A_m = X = x_1$, and $Y = x_1^{s_m/s_{m+1}}$, whence equality holds throughout only if, in addition, $x_{m+1} = Y^{1/(1-\alpha)} = x_1$ also. But, clearly, the equality holds if all the x 's are equal. This finishes the proof. \square



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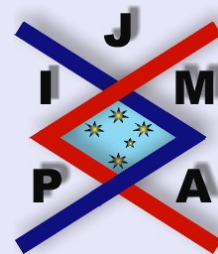
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References

- [1] G. BENNETT, Factorizing the classical inequalities, *Mem. Amer. Math. Soc.*, **120** (1996), no. 576, viii+130 pp.
- [2] G. BENNETT, An inequality for Hausdorff means, *Houston J. Math.*, **25**(4) (1996), 709–744.
- [3] G. BENNETT, Summability matrices and random walk, *Houston J. Math.*, **28**(4) (2002), 865–898.
- [4] T. CARLEMAN, Sur les fonctions quasi-analytiques, *Fifth Scand. Math. Congress* (1923), 181–196.
- [5] J.A. COCHRAN AND C.S. LEE, Inequalities related to Hardy's and Heinig's, *Math. Proc. Camb. Phil. Soc.*, **96**(1) (1984), 1–7.
- [6] G.H. HARDY, Notes on some points of the integral calculus (LX), *Messenger of Math.*, **54** (1925), 150–156.
- [7] G.H. HARDY, J.E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.
- [8] H. HEINIG, Some extensions of Hardy's inequality, *SIAM J. Math. Anal.*, **6** (1975), 698–713.
- [9] F. HOLLAND, On a mixed arithmetic-mean, geometric-mean inequality, *Math. Competitions*, **5** (1992), 60–64.
- [10] F. HOLLAND, A strengthening of the Carleman-Hardy-Pólya inequality, *Proc. Amer. Math. Soc.*, To appear.



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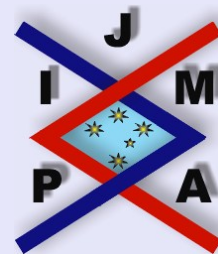
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- [11] M. JOHANSSON, L.E. PERSSON AND A. WEDESTIG, Carleman's inequality—history, proofs and some new generalizations, *J. Inequal. Pure & Appl. Math.*, **4**(3) (2003), Art. 53. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=291>].
- [12] K. KEDLAYA, Proof of a mixed arithmetic-mean, geometric-mean inequality, *Amer. Math. Monthly*, **101** (1994), 355–357.
- [13] K. KEDLAYA, A Weighted Mixed-Mean Inequality, *Amer. Math. Monthly*, **106** (1999), 355–358.
- [14] E.R. LOVE, Inequalities related to those of Hardy and of Cochran and Lee, *Math. Proc. Camb. Phil. Soc.*, **99**(1) (1986), 395–408.
- [15] E.R. LOVE, Inequalities related to Carleman's inequality, *Inequalities*, (Birmingham, 1987), 135–141, *Lecture Notes in Pure and Appl. Math.* **129**, Dekker, New York, 1991.
- [16] B. MOND AND J.E. PEČARIĆ, A mixed means inequality, *Austral. Math. Soc. Gaz.*, **23**(2) (1996) 67–70.
- [17] T.S. NANJUNDIAH, Sharpening of some classical inequalities, *Math Student*, **20** (1952), 24–25.
- [18] B. OPIC AND P. GURKA, Weighted inequalities for geometric means, *Proc. Amer. Math. Soc.*, **120** (1994), 771–779.



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