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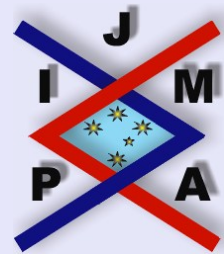
NEW INEQUALITIES ABOUT CONVEX FUNCTIONS

LAZHAR BOUGOFFA

Al-imam Muhammad Ibn Saud Islamic University
Faculty of Computer Science
Department of Mathematics
P.O. Box 84880, Riyadh 11681
Saudi Arabia.

EMail: bougoffa@hotmail.com

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Abstract

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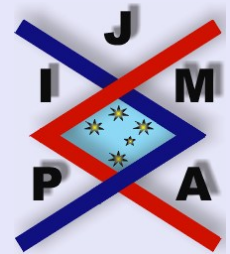


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Abstract

If f is a convex function and x_1, \dots, x_n or a_1, \dots, a_n lie in its domain the following inequalities are proved

$$\sum_{i=1}^n f(x_i) - f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{n-1}{n} \left[f\left(\frac{x_1 + x_2}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \right]$$

and

$$(n-1)[f(b_1) + \dots + f(b_n)] \leq n[f(a_1) + \dots + f(a_n) - f(a)],$$

where $a = \frac{a_1 + \dots + a_n}{n}$ and $b_i = \frac{na - a_i}{n-1}$, $i = 1, \dots, n$.

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1. Main Theorems

The well-known Jensen's inequality is given as follows [1]:

Theorem 1.1. *Let f be a convex function on an interval I and let w_1, \dots, w_n be nonnegative real numbers whose sum is 1. Then for all $x_1, \dots, x_n \in I$,*

$$(1.1) \quad w_1 f(x_1) + \dots + w_n f(x_n) \geq f(w_1 x_1 + \dots + w_n x_n).$$

Recall that a function f is said to be convex if for any $t \in [0, 1]$ and any x, y in the domain of f ,

$$(1.2) \quad t f(x) + (1 - t) f(y) \geq f(tx + (1 - t)y).$$

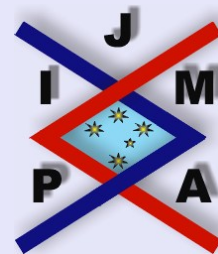
The aim of the present note is to establish new inequalities similar to the following known inequalities:

(Via Titu Andreescu (see [2, p. 6]))

$$\begin{aligned} f(x_1) + f(x_2) + f(x_3) + f\left(\frac{x_1 + x_2 + x_3}{3}\right) \\ \geq \frac{4}{3} \left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right) \right], \end{aligned}$$

where f is a convex function and x_1, x_2, x_3 lie in its domain, (Popoviciu inequality [3])

$$\sum_{i=1}^n f(x_i) + \frac{n}{n-2} f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{2}{n-2} \sum_{i < j} f\left(\frac{x_i + x_j}{2}\right),$$



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where f is a convex function on I and $x_1, \dots, x_n \in I$, and
(Generalized Popoviciu inequality)

$$(n-1)[f(b_1) + \dots + f(b_n)] \leq f(a_1) + \dots + f(a_n) + n(n-2)f(a),$$

where $a = \frac{a_1 + \dots + a_n}{n}$ and $b_i = \frac{na - a_i}{n-1}$, $i = 1, \dots, n$, and $a_1, \dots, a_n \in I$.

Our main results are given in the following theorems:

Theorem 1.2. *If f is a convex function and x_1, x_2, \dots, x_n lie in its domain, then*

$$(1.3) \quad \sum_{i=1}^n f(x_i) - f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{n-1}{n} \left[f\left(\frac{x_1 + x_2}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \right].$$

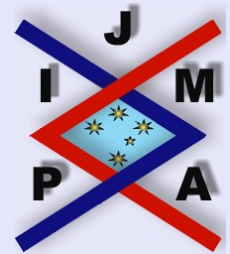
Proof. Using (1.2) with $t = \frac{1}{2}$, we obtain

$$(1.4) \quad f\left(\frac{x_1 + x_2}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \leq f(x_1) + f(x_2) + \dots + f(x_n).$$

In the summation on the right side of (1.4), the expression $\sum_{i=1}^n f(x_i)$ can be written as

$$\sum_{i=1}^n f(x_i) = \frac{n}{n-1} \sum_{i=1}^n f(x_i) - \frac{1}{n-1} \sum_{i=1}^n f(x_i),$$

$$\sum_{i=1}^n f(x_i) = \frac{n}{n-1} \left[\sum_{i=1}^n f(x_i) - \sum_{i=1}^n \frac{1}{n} f(x_i) \right].$$



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Replacing $\sum_{i=1}^n f(x_i)$ with the equivalent expression in (1.4),

$$f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \leq \frac{n}{n-1} \left[\sum_{i=1}^n f(x_i) - \sum_{i=1}^n \frac{1}{n} f(x_i) \right].$$

Hence, applying Jensen's inequality (1.1) to the right hand side of the above resulting inequality we get

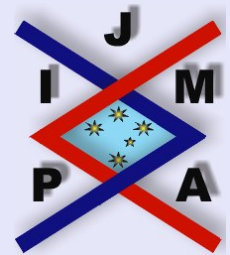
$$f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \leq \frac{n}{n-1} \left[\sum_{i=1}^n f(x_i) - f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \right],$$

and this concludes the proof. \square

Remark 1. Now we consider the simplest case of Theorem 1.2 for $n = 3$ to obtain the following variant of via Titu Andreescu [2]:

$$f(x_1) + f(x_2) + f(x_3) - f\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq \frac{2}{3} \left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right) \right].$$

The variant of the generalized Popoviciu inequality is given in the following theorem.



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Theorem 1.3. If f is a convex function and a_1, \dots, a_n lie in its domain, then

$$(1.5) \quad (n-1)[f(b_1) + \dots + f(b_n)] \leq n[f(a_1) + \dots + f(a_n) - f(a)],$$

where $a = \frac{a_1 + \dots + a_n}{n}$ and $b_i = \frac{na - a_i}{n-1}$, $i = 1, \dots, n$.

Proof. By using the Jensen inequality (1.1),

$$f(b_1) + \dots + f(b_n) \leq f(a_1) + \dots + f(a_n),$$

and so,

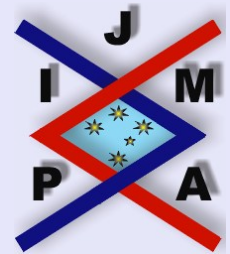
$$\begin{aligned} f(b_1) + \dots + f(b_n) \\ \leq \frac{n}{n-1}[f(a_1) + \dots + f(a_n)] - \frac{1}{n-1}[f(a_1) + \dots + f(a_n)], \end{aligned}$$

or

$$\begin{aligned} f(b_1) + \dots + f(b_n) \\ \leq \frac{n}{n-1}[f(a_1) + \dots + f(a_n)] - \frac{n}{n-1} \left[\frac{1}{n}f(a_1) + \dots + \frac{1}{n}f(a_n) \right], \end{aligned}$$

and so

$$(1.6) \quad \begin{aligned} f(b_1) + \dots + f(b_n) \\ \leq \frac{n}{n-1} \left[f(a_1) + \dots + f(a_n) - \left(\frac{1}{n}f(a_1) + \dots + \frac{1}{n}f(a_n) \right) \right]. \end{aligned}$$



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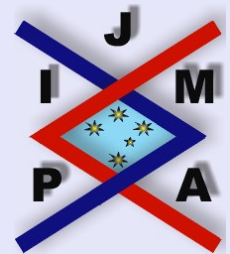
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Hence, applying Jensen's inequality (1.1) to the right hand side of (1.6) we get

$$f(b_1) + \cdots + f(b_n) \leq \frac{n}{n-1} \left[f(a_1) + \cdots + f(a_n) - f\left(\frac{a_1 + \cdots + a_n}{n}\right) \right],$$

and this concludes the proof. \square



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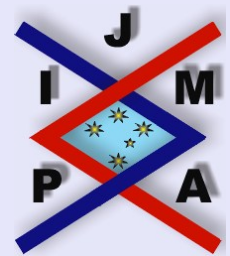
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