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REFINEMENTS AND SHARPENINGS OF SOME DOUBLE INEQUALITIES FOR BOUNDING THE GAMMA FUNCTION

BAI-NI GUO, YING-JIE ZHANG, AND FENG QI

SCHOOL OF MATHEMATICS AND INFORMATICS
HENAN POLYTECHNIC UNIVERSITY
JIAOZUO CITY, HENAN PROVINCE
454010, CHINA
bai.ni.guo@gmail.com

DEPARTMENT OF MATHEMATICS
JIAOZUO UNIVERSITY
JIAOZUO CITY, HENAN PROVINCE
454003, CHINA

RESEARCH INSTITUTE OF MATHEMATICAL INEQUALITY THEORY
HENAN POLYTECHNIC UNIVERSITY
JIAOZUO CITY, HENAN PROVINCE
454010, CHINA
qifeng618@hotmail.com
URL: http://qifeng618.spaces.live.com/

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ABSTRACT. In this paper, some sharp inequalities for bounding the gamma function $\Gamma(x)$ and the ratio of two gamma functions are established. From these, several known results are recovered, refined, extended and generalized simply and elegantly.

Key words and phrases: Inequality, Refinement, Sharpening, Generalization, Kečlić-Vasić-Alzer's double inequalities.

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In [4], it was proved that the function

(1)
$$f(x)=\frac{\ln\Gamma(x+1)}{x\ln x}$$

is strictly increasing from $(1,\infty)$ onto $(1-\gamma,1)$, where γ is Euler-Mascheroni's constant. In particular, for $x\in(1,\infty)$,

(2)
$$x^{(1-\gamma)x-1}<\Gamma(x)<x^{x-1}.$$

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In [1, Theorem 2], inequality (2) was extended and sharpened: If $x \in (0,1)$, then

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma} \tag{3}$$

with the best possible constants $\alpha = 1-\gamma$ and $\beta = \frac{1}{2}\big(\frac{\pi^2}{6}-\gamma\big)$. If $x \in (1,\infty)$, then inequality (3) holds with the best possible constants $\alpha = \frac{1}{2}\big(\frac{\pi^2}{6}-\gamma\big)$ and $\beta = 1$.

In [8], by using the convolution theorem for Laplace transforms and other techniques, inequalities (2) and (3) were refined: The double inequality

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}} \tag{4}$$

holds for $x > 1$ and the constants γ and $\frac{1}{2}$ are the best possible. For $0 < x < 1$, the left-hand inequality in (4) still holds, but the right-hand inequality in (4) reverses.

Remark 1. The double inequality (4) can be verified simply as follows: In [3], the function

$$\theta(x) = x[\ln x - \psi(x)] \tag{5}$$

was proved to be decreasing and convex in $(0,\infty)$ with $\theta(1) = \gamma$ and two limits $\lim_{x\to 0^+}\theta(x) = 1$ and $\lim_{x\to\infty}\theta(x) = \frac{1}{2}$. Since the function $g_\alpha(x) = \frac{e^x\Gamma(x)}{x^{x-\alpha}}$ for $x > 0$ satisfies $\frac{xg'_\alpha(x)}{g_\alpha(x)} = x[\psi(x) - \ln x] + \alpha$, it increases for $\alpha \ge 1$, decreases for $\alpha \le \frac{1}{2}$, and has a unique minimum for $\frac{1}{2} < \alpha < 1$ in $(0,\infty)$. This implies that the function $g_\alpha(x)$ decreases in $(0, x_0)$ and increases in (x_0,∞) for $\alpha = x_0[\ln x_0 - \psi(x_0)]$ and all $x_0 \in (0,\infty)$. Hence, taking $x_0 = 1$ yields that $\alpha = \gamma$ and $g_\gamma(x)$ decreases in $(0,1)$ and increases in $(1,\infty)$, and taking $\alpha = \frac{1}{2}$ gives that the function $g_{1/2}(x)$ is decreasing in $(0,\infty)$. By virtue of $g_\alpha(1) = e$, the double inequality (4) follows.

The first main result of this paper is the following theorem which can be regarded as a generalization of inequalities (2), (3) and (4).

Theorem 2. *Let a be a positive number. Then the function $\frac{e^x\Gamma(x)}{x^{x-a[\ln a-\psi(a)]}}$ is decreasing in $(0,a]$ and increasing in $[a,\infty)$, and the function $\frac{e^x\Gamma(x)}{x^{x-b}}$ in $(0,\infty)$ is increasing if and only if $b \ge 1$ and decreasing if and only if $b \le \frac{1}{2}$.*

Proof. This follows from careful observation of the arguments in Remark 1. □

For $a > 0$ and $b > 0$ with $a \ne b$, the mean

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{6}$$

is called the identric or exponential mean. See [9] and related references therein.

As direct consequences of Theorem 2, several sharp inequalities related to the identric mean and the ratio of gamma functions are established as follows.

Theorem 3. *For $y > x \ge 1$,*

$$\frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-\gamma}}{y^{y-\gamma}}e^{y-x} \quad \text{or} \quad [I(x,y)]^{y-x} < \left(\frac{y}{x}\right)^{\gamma}\frac{\Gamma(y)}{\Gamma(x)}. \tag{7}$$

If $1 \ge y > x > 0$, inequality (7) reverses.

For $y > x > 0$, inequality

$$\frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-b}}{y^{y-b}}e^{y-x} \quad \text{or} \quad [I(x,y)]^{y-x} < \left(\frac{y}{x}\right)^{b}\frac{\Gamma(y)}{\Gamma(x)} \tag{8}$$

holds if and only if $b \ge 1$. The reversed inequality (8) is valid if and only if $b \le \frac{1}{2}$.

Proof. Letting $a = 1$ in Theorem 2 gives that the function $\frac{e^x\Gamma(x)}{x^{x-\gamma}}$ is decreasing in $(0,1]$ and increasing in $[1,\infty)$. Thus, for $y > x \ge 1$,

$$\frac{e^x\Gamma(x)}{x^{x-\gamma}} < \frac{e^y\Gamma(y)}{y^{y-\gamma}}. \tag{9}$$

Rearranging (9) leads to the inequalities in (7).

The rest of the proofs are similar, so we shall omit them. □

Remark 4. The inequalities in (7) and (8) have been obtained in [7] and [2, Theorem 4]. However, Theorem 3 provides an alternative and concise proof of Kečlić-Vasić-Alzer's double inequalities in [2, 7]. In [5, 6], several new inequalities similar to (7) and (8) were presented.

The third main results of this paper are refinements and sharpenings of the double inequalities (2), (3) and (4), which are stated below.

Theorem 5. *The function*

$$h(x) = \frac{e^x\Gamma(x)}{x^{x[1-\ln x+\psi(x)]}} \tag{10}$$

in $(0,\infty)$ *has a unique maximum* e *at* $x = 1$*, with the limits*

$$\lim_{x\to 0^+} h(x) = 1 \quad \textit{and} \quad \lim_{x\to\infty} h(x) = \sqrt{2\pi}\,. \tag{11}$$

Consequently, sharp double inequalities

$$\frac{x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) \le \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}} \tag{12}$$

in $(0,1]$ *and*

$$\frac{\sqrt{2\pi}\, x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) \le \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}} \tag{13}$$

in $[1,\infty)$ *are valid.*

Proof. Direct calculation yields

$$h'(x) = [\ln x - \psi(x) - x\psi'^{x} x^{x[\ln x-\psi(x)-1]}\Gamma(x)\ln x. \tag{14}$$

Since the factor $x\psi'(x) + \psi(x) - \ln x - 1 = -\theta'(x)$ and $\theta(x)$ is decreasing in $(0,\infty)$, the function $h(x)$ has a unique maximum e at $x = 1$.

The second limit in (11) follows from standard arguments by using the following two well known formulas: As $x \to \infty$,

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + \frac{1}{12x} + O\left(\frac{1}{x}\right), \tag{15}$$

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^2}\right). \tag{16}$$

Direct computation gives

$$\lim_{x\to 0^+} \ln h(x) = \lim_{x\to 0^+} [\ln\Gamma(x) - x\psi(x)\ln x] = 0 \tag{17}$$

by utilizing the following two well known formulas

$$-\ln\Gamma(x) = \ln x + \gamma x + \sum_{k=1}^{\infty}\left[\ln\left(1 + \frac{x}{k}\right) - \frac{x}{k}\right] \tag{18}$$

and

$$\psi(x) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{x+k}\right) \tag{19}$$

for $x > 0$. The proof is complete. □

Remark 6. The graph in Figure 1 plotted by MATHEMATICA 5.2 shows that the left hand sides

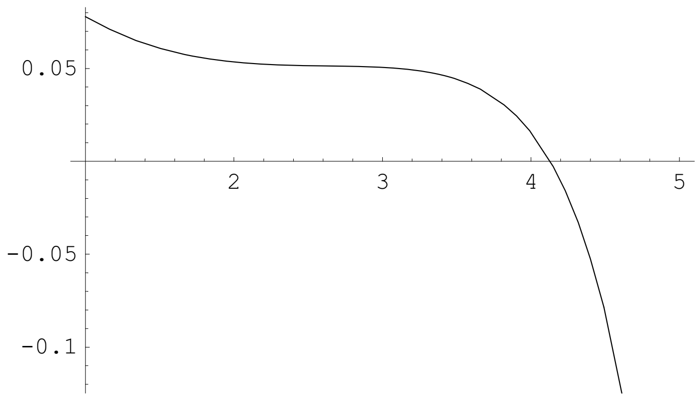

Figure 1: Graph of $\frac{x^{x-\gamma}}{e^{x-1}} - \frac{\sqrt{2\pi}\, x^{x[1-\ln x+\psi(x)]}}{e^x}$ *in* $(1,5)$

in double inequalities (4) and (13) for $x > 1$ do not include each other and that the lower bound in (13) is better than the one in (4) when $x > 1$ is large enough.

As discussed in Remark 1, the double inequality $\frac{1}{2} < x[\ln x - \psi(x)] < 1$ in $(0,\infty)$ clearly holds. Therefore, the upper bounds in (12) and (13) are better than the corresponding one in (4).

Theorem 7. *Inequality*

$$I(x,y) < \left\{\frac{x^{x[\ln x-\psi(x)]}\Gamma(x)}{y^{y[\ln y-\psi(y)]}\Gamma(y)}\right\}^{1/(x-y)} \tag{20}$$

holds true for $x \ge 1$ *and* $y \ge 1$ *with* $x \ne y$. *If* $0 < x \le 1$ *and* $0 < y \le 1$ *with* $x \ne y$, *inequality* (20) *is reversed.*

Proof. From Theorem 5, it is clear that the function $h(x)$ is decreasing in $[1,\infty)$ and increasing in $(0,1]$. A similar argument to the proof of Theorem 3 straightforwardly leads to inequality (20) and its reversed version. □

Remark 8. The inequality (20) is better than those in (7), since the function

$$q(t) \triangleq t^{t[\psi(t)-\ln t]-\gamma} \tag{21}$$

is decreasing in $(0,\infty)$ with $q(1) = 1$ and $\lim_{t\to0^+} q(t) = \infty$, which is shown by the graph of $q(t)$, plotted by MATHEMATICA 5.2.

It is conjectured that the function $q(t)$ is logarithmically completely monotonic in $(0,\infty)$.

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