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UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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ABSTRACT. In this paper we consider some integral operators and we determine conditions for the univalence of these integral operators.

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1. INTRODUCTION

Let $U = \{z \in C : |z| < 1\}$ be the unit disc in the complex plane. The class A and the class S are defined in [2]: let A be the class of functions $f(z) = z + a_2 z^2 + \cdots$, which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$; let S the class of the functions $f \in A$ which are univalent in U.

In [7] is defined the class $S(\alpha)$. For $0 < \alpha \le 2$, let $S(\alpha)$ denote the class of functions $f \in A$ which satisfy the conditions:

$$f(z) \neq 0 \qquad \text{for } 0 < |z| < 1 \tag{1.1}$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \le \alpha \tag{1.2}$$

for all $z \in U$.

In [7] is proved the next result. For $0 < \alpha \le 2$, the functions $f \in S(\alpha)$ are univalent.

In this work, we consider the integral operators

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\, du \right]^{\frac{1}{\alpha}} \tag{1.3}$$

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and

$$H_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{\frac{1}{\alpha}} \tag{1.4}$$

for $g(z) \in S,\ h(z) \in S$ and for some $\alpha,\ \gamma \in C$.

Kim - Merkes [1] studied the integral operator

$$F_{\gamma}(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\gamma} du \tag{1.5}$$

and obtained the following result

Theorem 1.1. *If the function $h(z)$ belongs to the class S, then for any complex number $\gamma,\ \ |\gamma| \le \frac{1}{4}$, the function $F_{\gamma}(z)$ defined by (1.5) is in the class S.*

2. Preliminary Results

In order to prove our main results we will use the lemma due to N.N. Pascu [4] presented in this section.

Lemma 2.1. *Let the function $f \in A$ and α a complex number,* $\operatorname{Re}\alpha > 0$. *If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.1}$$

for all $z \in U$, then for all complex numbers $\beta,\ \operatorname{Re}\beta \ge \operatorname{Re}\alpha$ the function

$$F_{\beta}(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is regular and univalent in U.

3. Main Results

Theorem 3.1. *Let α be a complex number,* $\operatorname{Re}\ \alpha \ge 0$ *and the function $g \in S,\ \ g(z) = z + a_2 z^2 + \cdots$. If*

$$|\alpha - 1| \le \frac{\operatorname{Re}\ \alpha}{4} \qquad \textit{for } \operatorname{Re}\ \alpha \in (0,1) \tag{j_1}$$

or

$$|\alpha - 1| \le \frac{1}{4} \qquad \textit{for } \operatorname{Re}\ \alpha \in [1,\infty), \tag{j_2}$$

then the function

$$G_{\alpha}(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\, du\right]^{\frac{1}{\alpha}} \tag{3.1}$$

is in the class S.

Proof. From (3.1) we have

$$G_{\alpha}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{3.2}$$

The function $g(z)$ is regular and univalent, hence $\frac{g(z)}{z} \ne 0$ for all $z \in U$. We can choose the regular branch of the function $\left[\frac{g(z)}{z}\right]^{\alpha-1}$ to be equal to 1 at the origin.

Let us consider the regular function in U, given by

$$p(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha-1} du. \tag{3.3}$$

Because $g \in S$, we obtain

$$\left|\frac{z\,g'(z)}{g(z)}\right| \le \frac{1+|z|}{1-|z|} \tag{3.4}$$

for all $z \in U$.

We have

$$\begin{aligned} \frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,p''(z)}{p'(z)}\right| &= \frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,g'(z)}{g(z)}-1\right| \\ &\le \frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right). \end{aligned} \tag{3.5}$$

From (3.5) and (3.4) we obtain

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,p''(z)}{p'(z)}\right| \le \frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\frac{2}{1-|z|}. \tag{3.6}$$

Now, we consider the cases

$i_1)\quad 0 < \mathrm{Re}\,\alpha < 1.$

The function

$$s : (0,1) \to \Re, \quad s(x) = 1 - a^{2x} \quad (0 < a < 1)$$

is a increasing function and for $a = |z|, \quad z \in U$, we obtain

$$1 - |z|^{2\,\mathrm{Re}\,\alpha} \le 1 - |z|^2 \tag{3.7}$$

for all $z \in U$.

From (3.6) and (3.7), we have

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{4\,|\alpha-1|}{\mathrm{Re}\,\alpha} \tag{3.8}$$

for all $z \in U$.

Using the condition (j_1) and (3.8) we get

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le 1 \tag{3.9}$$

for all $z \in U$.

$i_2)\quad \mathrm{Re}\,\alpha \ge 1.$

We observe that the function

$$q : [1,\infty) \to \Re, \quad q(x) = \frac{1-a^{2x}}{x} \quad (0 < a < 1)$$

is a decreasing function, and that, if we take $a = |z|,\ z \in U$, then

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \le 1 - |z|^2 \tag{3.10}$$

for all $z \in U$.

From (3.6) and (3.10) we obtain

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le 4\,|\alpha-1|. \tag{3.11}$$

From (3.11) and (j_2), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le 1 \tag{3.12}$$

for all $z \in U$.

Using (3.9), (3.12) and because $p'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $G_\alpha(z)$ is in the class S. □

Theorem 3.2. *If α is a real number, $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$ and the function $g \in S(\alpha)$, then the function*

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\, du\right]^{\frac{1}{\alpha}} \tag{3.13}$$

is in the class S.

Proof. If $g \in S(\alpha)$, then $g \in S$ and by Theorem 3.1 for $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$, we obtain the function $G_\alpha(z)$ in the class S. □

Theorem 3.3. *Let α, γ be a complex numbers and the function $h \in S,\ h(z) = z + a_2 z^2 + \cdots$. If*

$$|\gamma| \le \frac{\operatorname{Re}\alpha}{4} \quad \textit{for} \quad \operatorname{Re}\alpha \in (0,1) \tag{p_1}$$

or

$$|\gamma| \le \frac{1}{4} \quad \textit{for} \quad \operatorname{Re}\alpha \in [1,\infty) \tag{p_2}$$

then the function

$$H_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{\frac{1}{\alpha}} \tag{3.14}$$

is regular and univalent in U.

Proof. Let us consider the regular function in U, defined by

$$f(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\gamma} du. \tag{3.15}$$

For the function $h \in S$, we obtain

$$\left|\frac{z\,h'(z)}{h(z)}\right| \le \frac{1+|z|}{1-|z|} \tag{3.16}$$

for all $z \in U$.

We obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z\,f''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\gamma|\left(\left|\frac{zh'(z)}{h(z)}\right|+1\right). \tag{3.17}$$

From (3.17) and (3.16), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z\,f''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\gamma|\frac{2}{1-|z|} \tag{3.18}$$

We consider the cases

$j_1)\quad 0 < \operatorname{Re}\alpha < 1.$

In this case we obtain

$$1-|z|^{2\operatorname{Re}\alpha} \le 1-|z|^2 \tag{3.19}$$

for all $z \in U$.

From (3.18) and (3.19), we get

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{4|\gamma|}{\operatorname{Re}\alpha} \tag{3.20}$$

for all $z \in U$.

By (3.20) and (p_1) we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{3.21}$$

for all $z \in U$.

j_2) $\operatorname{Re}\alpha \ge 1$.

For this case we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le 1-|z|^2 \tag{3.22}$$

for all $z \in U$.

From (3.18) and (3.22) we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 4|\gamma|. \tag{3.23}$$

From (3.23) and (p_2), we get

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{3.24}$$

for all $z \in U$.

From (3.21), (3.24) and because $f'(z) = \left(\frac{h(z)}{z}\right)^{\gamma}$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $H_{\alpha,\gamma}(z)$ is in the class S. □

Remark 3.4. For $\alpha = 1$, from Theorem 3.3 we obtain Theorem 1.1, the result due to Kim-Merkes.

Theorem 3.5. *Let γ be a complex number and the function $h \in S(a)$.*
If

$$|\gamma| \le \frac{\alpha}{4} \quad \textit{for} \quad \alpha \in (0,1) \tag{3.25}$$

or

$$|\gamma| \le \frac{1}{4} \quad \textit{for} \quad \alpha \in [1,2] \tag{3.26}$$

then the function $H_{\alpha,\gamma}(z)$ defined by (3.14) is in the class S.

Proof. Because $h(z) \in S(\alpha)$, $0 < \alpha \le 2$, then $h(z) \in S$ and by Theorem 3.3 the function $H_{\alpha,\gamma}(z)$ belongs to the class S. □

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