journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 4, Article 102, 12 pp.

LOGARITHMIC FUNCTIONAL MEAN IN CONVEX ANALYSIS

MUSTAPHA RAÏSSOULI

AFA TEAM, AFACSI LABORATORY
MY ISMAÏL UNIVERSITY,
FACULTY OF SCIENCES, P.O. BOX 11201
MEKNÈS, MOROCCO
raissouli_10@hotmail.com

Received 11 June, 2008; accepted 29 April, 2009
Communicated by P.S. Bullen

ABSTRACT. In this paper, we present various functional means in the sense of convex analysis. In particular, a logarithmic mean involving convex functionals, extending the scalar one, is introduced. In the quadratic case, our functional approach implies immediately that of positive operators. Some examples, illustrating theoretical results and showing the interest of our functional approach, are discussed.

Key words and phrases: Convex analysis, operator and functional means.

2000 *Mathematics Subject Classification.* 46A20, 52A41, 49-XX, 52-XX.

1. INTRODUCTION

Recently, many authors have been interested in the construction of means involving convex functionals and extending that of scalar and operator ones. The original idea, due to Atteia-Raïssouli [1], comes from the fact that the Legendre-Fenchel conjugate operation can be considered as an inverse in the sense of convex analysis. This interpretation has allowed them to introduce a functional duality, with which they have constructed for the first time, the convex geometric functional mean, which in the quadratic case, immediately gives the operator result already discussed by some authors, [3, 6, 7, 9]. After this, several works [5, 8, 13, 14, 15, 16, 17, 18, 19] proved that the theory of functional means contains that of means for positive operators.

In this paper we introduce a class of functional means in convex analysis. To convey the key idea to the reader, we wish to briefly describe our aim in the following. The logarithmic mean of two positive reals a and b is known as

$$\mathbb{L}(a,b) = \frac{a-b}{\ln a - \ln b} \text{ if } a \neq b, \quad \mathbb{L}(a,a) = a,$$

or alternatively, in integral form as

$$(\mathbb{L}(a,b))^{-1} = \int_0^{+\infty} \frac{dt}{(a+t)(b+t)}.$$

171-08

The extension of the logarithmic mean from the scalar case to the functional one is not obvious and appears to be interesting: what should be the analogue of $\mathbb{L}(a,b)$ when the variables a and b are convex functionals? The functional logarithm in convex analysis has already been introduced in [14], but it is not sufficient since the product (resp. quotient) of two convex functionals, extending that of operators, has not yet been covered. This is where the difficulty lies in extending the logarithmic mean from the two above scalar forms.

A third representation of $\mathbb{L}(a,b)$ is given by the convex form

$$(\mathbb{L}(a,b))^{-1} = \int_0^1 (t \cdot a + (1-t) \cdot b)^{-1} dt,$$

whose importance stems from the fact that it does not contain any product nor quotient of scalars, but only an inverse which, as already mentioned, has been extended to convex functionals. For this, we suggest that a reasonable analogue of $\mathbb{L}(a,b)$ involving convex functionals f and g is

$$\mathbb{L}(f,g) = \left(\int_0^1 (t \cdot f + (1-t) \cdot g)^* dt \right)^*,$$

where $*$ denotes the conjugate operation defined, for a functional $\Phi \in \widetilde{\mathbb{R}}^E$, by the relationship

$$\Phi^*(u^*) = \sup_{u \in E} \{ \langle u^*, u \rangle - \Phi(u) \}.$$

In the quadratic case, i.e., if

$$f(u) = f_A(u) := \frac{1}{2} \langle Au, u \rangle, \quad g(u) = f_B(u) := \frac{1}{2} \langle Bu, u \rangle,$$

for all $u \in E$, where $A, B : E \longrightarrow E^*$ are two positive invertible operators, then we obtain immediately a convex form of the logarithmic operator mean of A and B given by

$$\mathbb{L}(A,B) = \left(\int_0^1 (t \cdot A + (1-t) \cdot B)^{-1} dt \right)^{-1}.$$

This paper is divided into five parts and organized as follows: Section 2 contains some basic notions of convex analysis that are needed throughout the paper. In Section 3, we introduce the logarithmic mean of two convex functionals and we study its properties. Section 4 is devoted to the intermediary functional means constructed from the arithmetic, logarithmic and harmonic ones. Finally, in Section 5 we present the logarithmic mean of several functional variables from which we deduce another intermediary mean, called the arithmetic-logarithmic-harmonic functional mean. In the quadratic case, the above definitions and results immediately give those of positive operators.

2. PRELIMINARY RESULTS

In this section, we recall some standard notations and results in convex analysis which are needed in the sequel. For further details, the reader can consult for instance [2, 4, 10].

Let E be a real normed space (reflexive Banach when it is necessary), E^* its topological dual, and $\langle \cdot, \cdot \rangle$ the duality bracket between E and E^*.

If we denote by $\overline{\mathbb{R}}^E$ the space of all functions defined from E into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we can extend the structure of $\mathbb{R}$ on $\overline{\mathbb{R}}$ by setting

$$\forall x \in \overline{\mathbb{R}}, \quad -\infty \le x \le +\infty, \quad (+\infty) + x = +\infty, \quad 0 \cdot (+\infty) = +\infty.$$

With this, the space $\overline{\mathbb{R}}^E$ is equipped with the partial ordering relation defined by

$$\forall f, g \in \overline{\mathbb{R}}^E, \quad f \le g \iff \forall u \in E \quad f(u) \le g(u).$$

Given a functional $f : E \to \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, we denote by f^* the Legendre-Fenchel conjugate of f defined by

$$\forall u^* \in E^* \qquad f^*(u^*) = \sup_{u \in E}\{\langle u, u^*\rangle - f(u)\}.$$

It is clear that, if $f \le g$ then $g^* \le f^*$.

We notice that, if E is a complex normed space, the conjugate operation can be replaced by the extended one,

$$\forall u^* \in E^* \quad f^*(u^*) = \sup_{u \in E}\{\mathrm{Re}\,\langle u^*, u\rangle - f(u)\},$$

where $\mathrm{Re}\,\langle u^*, u\rangle$ denotes the real part of the complex number $\langle u^*, u\rangle$.

In what follows, we restrict ourselves to the case of real normed spaces since the complex one can be stated in a similar manner.

Let $f \in \widetilde{\mathbb{R}}^E$ and $\lambda > 0$ be a real. We define the functionals $\lambda \cdot f$ and $f \cdot \lambda$ by

$$\forall u \in E, \quad (\lambda \cdot f)(u) = \lambda \cdot f(u) \;\text{ and }\; (f \cdot \lambda)(u) = \lambda \cdot f\left(\frac{u}{\lambda}\right).$$

With this, it is not hard to see that

$$(\lambda \cdot f)^* = f^* \cdot \lambda \;\text{ and }\; (f \cdot \lambda)^* = \lambda \cdot f^*.$$

The bi-conjugate of f is the functional $f^{**} : E \longrightarrow \widetilde{\mathbb{R}}$ defined as follows

$$\forall u \in E \quad f^{**}(u) := (f^*)^*(u) = \sup_{u^* \in E^*}\{\langle u^*, u\rangle - f^*(u^*)\}.$$

It is well-known that, $f^{**} \le f$ and, $f^{**} = f$ if and only if $f \in \Gamma_0(E)$, where $\Gamma_0(E)$ is the cone of lower semi-continuous convex functionals from E into $\mathbb{R} \cup \{+\infty\}$ not identically equal to $+\infty$. Analogously, we can define $f^{***} : E^* \longrightarrow \widetilde{\mathbb{R}}$ which satisfies $f^{***} = f^*$, and thus $f^* \in \Gamma_0(E^*)$ for all $f \in \widetilde{\mathbb{R}}^E$.

An important and typical example of a $\Gamma_0(E)-$functional is f_A defined by

$$\forall u \in E \qquad f_A(u) = \frac{1}{2}\langle Au, u\rangle,$$

where $A : E \longrightarrow E^*$ is a bounded linear positive operator. We say that f_A is quadratic in the sense that $f(t \cdot u) = t^2 f(u)$ for all $u \in E$ and $t \in \mathbb{R}$. It is easy to see that the conjugate operation preserves the quadratic character. If, moreover, A is invertible then f_A^* has the explicit form

$$\forall u^* \in E^* \qquad f_A^*(u^*) = \frac{1}{2}\left\langle A^{-1}u^*, u^*\right\rangle.$$

That is, $f_A^* = f_{A^{-1}}$ and so, as already observed, the conjugate operation can be considered as a reasonable extension of the inverse operator in the sense of convex analysis.

Now, let us recall that, for all $f, g \in \widetilde{\mathbb{R}}^E$ and $\alpha \in]0,1[$,

$$(\alpha f + (1-\alpha)g)^* \le \alpha f^* + (1-\alpha)g^*, \tag{2.1}$$

i.e., the map $f \longmapsto f^*$ is convex with respect to the point-wise ordering on $\widetilde{\mathbb{R}}^E$.

In the quadratic case, i.e., if $f = f_A$ and $g = f_B$, and $A, B : E \longrightarrow E^*$ are positive invertible operators, then the above inequality immediately implies,

$$\forall \alpha \in]0,1[\qquad (\alpha \cdot A + (1-\alpha) \cdot B)^{-1} \le \alpha \cdot A^{-1} + (1-\alpha) \cdot B^{-1},$$

which, without the conjugate operation, is not directly obvious, see [11] for example.

A convex-integral version of inequality (2.1) is given in the following result.

Lemma 2.1. *Let Ω be a nonempty subset of $\mathbb{R}^m$, $F : \Omega \times E \longrightarrow \widetilde{\mathbb{R}}$ and $\psi : \Omega \longrightarrow [0, +\infty[$ such that $\int_\Omega \psi(t)dt = 1$. If we put*

$$\forall u \in E \quad \Phi(u) = \int_\Omega \psi(t)F(t,u)dt,$$

then the conjugate functional $\Phi^* : E^* \longrightarrow \widetilde{\mathbb{R}}$ of Φ satisfies

$$\forall u^* \in E^* \quad \Phi^*(u^*) \leq \int_\Omega \psi(t)F^*(t,u^*)dt,$$

where

$$F^*(t,u^*) = \sup_{u\in E}\{ \langle u^*, u\rangle - F(t,u)\}.$$

Proof. For $m = 1,\ \Omega =]a, b[\subset \mathbb{R}$, this lemma is proved in [16]. For $m \geq 2$, the same result is achieved by using arguments analogous to those in [16]. □

Finally, for $f, g \in \widetilde{\mathbb{R}}^E$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, the arithmetic and harmonic functional means of f and g are respectively defined by, [1]

$$\mathbb{A}(f,g) = \frac{f+g}{2}, \quad \mathbb{H}(f,g) = \left(\frac{1}{2}f^* + \frac{1}{2}g^*\right)^*. \tag{2.2}$$

Clearly, $\mathbb{H}(f,g) \in \Gamma_0(E)$ and, if $f, g \in \Gamma_0(E)$ then so is $\mathbb{A}(f,g)$. Moreover, (2.1) immediately gives the arithmetic-harmonic mean inequality

$$\forall f, g \in \widetilde{\mathbb{R}}^E \quad \mathbb{H}(f,g) \leq \mathbb{A}(f,g).$$

3. LOGARITHMIC FUNCTIONAL MEAN

As mentioned before, the fundamental goal of this section is to introduce the logarithm mean of two convex functionals. Such functional means extend available results for that of positive real numbers.

Definition 3.1. Let $f, g \in \Gamma_0(E)$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. We put

$$\mathbb{L}(f,g) = \left(\int_0^1 (t \cdot f + (1-t) \cdot g)^* dt\right)^*,$$

which will be called the logarithmic functional mean of f and g in the sense of convex analysis.

The fact that f, g belong to $\Gamma_0(E)$ is not the only way to define $\mathbb{L}(f,g)$. The logarithmic mean of f and g can be defined by the above formulae for all $f, g \in \widetilde{\mathbb{R}}^E$. However, in order to simplify the presentation for the reader, we assume that $f, g \in \Gamma_0(E)$.

The elementary properties of $\mathbb{L}(f,g)$ are summarized in the following.

Proposition 3.1. *Let $f, g \in \Gamma_0(E)$. The following statements hold true.*

(1) $\mathbb{L}(f,g) \in \Gamma_0(E),\ \mathbb{L}(f,f) = f,\ \mathbb{L}(f,g) = \mathbb{L}(g,f)$.
(2) $\mathbb{L}(\lambda \cdot f, \lambda \cdot g) = \lambda \cdot \mathbb{L}(f,g)$ *and* $\mathbb{L}(f \cdot \lambda, g \cdot \lambda) = \mathbb{L}(f,g) \cdot \lambda$, *for all* $\lambda > 0$.
(3) $\mathbb{L}(f+a, g+b) = \mathbb{L}(f,g) + \mathbb{A}(a,b)$, *for all* $a, b \in \mathbb{R}$.

Proof. It is immediate from the definition with the properties of the conjugate operation. □

Proposition 3.2. *Let f_1, f_2, g_1, g_2 in $\Gamma_0(E)$ such that $f_1 \leq f_2$ and $g_1 \leq g_2$. Then*

$$\mathbb{L}(f_1, g_1) \leq \mathbb{L}(f_2, g_2).$$

In particular, for all $f, g \in \Gamma_0(E)$ one has

$$(\inf(f,g))^{**} \leq \mathbb{L}(f,g) \leq \sup(f,g).$$

Proof. It is immediate from the properties of the conjugate operation stated in Section 2. □

Proposition 3.3. *Let $f, g \in \Gamma_0(E)$. Then the following arithmetic-logarithmic-harmonic mean inequality holds*

$$\mathbb{H}(f,g) \le \mathbb{L}(f,g) \le \mathbb{A}(f,g),$$

where $\mathbb{A}(f,g)$ and $\mathbb{H}(f,g)$ are respectively defined by (2.2).

Proof. By Lemma 2.1, we obtain

$$\mathbb{L}(f,g) \le \int_0^1 (t\cdot f + (1-t)\cdot g)^{**}dt \le \int_0^1 (t\cdot f + (1-t)\cdot g)dt = \frac{f+g}{2},$$

which gives the right inequality.

To prove the left one, we use (2.1) to write

$$\int_0^1 (t\cdot f + (1-t)\cdot g)^{*}dt \le \int_0^1 (t\cdot f^* + (1-t)\cdot g^*)dt = \frac{f^*+g^*}{2},$$

which, by taking the polar of the two members, gives

$$\left(\frac{f^*+g^*}{2}\right)^* \le \left(\int_0^1 (t\cdot f + (1-t)\cdot g)^{*}dt\right)^*,$$

thus proving the desired result. □

Corollary 3.4. *Let $f, g \in \Gamma_0(E)$ such that* $\operatorname{dom} f = \operatorname{dom} g$. *Then*

$$\operatorname{dom} \mathbb{H}(f,g) = \operatorname{dom} \mathbb{L}(f,g) = \operatorname{dom} \mathbb{A}(f,g) = \operatorname{dom} f.$$

Proof. In [5], the authors proved that $\operatorname{dom} f = \operatorname{dom} g$ if and only if

$$\operatorname{dom} \mathbb{H}(f,g) = \operatorname{dom} \mathbb{A}(f,g).$$

This, with the latter proposition, and the fact that

$$\operatorname{dom} \mathbb{A}(f,g) = \operatorname{dom} f \cap \operatorname{dom} g,$$

implies the desired result. □

We notice that the above hypothesis $\operatorname{dom} f = \operatorname{dom} g$, also assumed below, is not a restriction since it can be omitted with regularization. In this sense, the reader can consult [5] for similar examples of regularization.

Proposition 3.5. *Let $f, g \in \Gamma_0(E)$. If f and g are quadratic, then so is $\mathbb{L}(f,g)$. Moreover, if $f = f_A$ and $g = f_B$, where A and B are two positive invertible operators then*

$$\mathbb{L}(f,g) = f_{\mathbb{L}(A,B)},$$

with

$$\mathbb{L}(A,B) = \left(\int_0^1 (t\cdot A + (1-t)\cdot B)^{-1}dt\right)^{-1}. \tag{3.1}$$

Proof. The result comes from the fact that

$$t\cdot f_A + (1-t)\cdot f_B = f_{t\cdot A+(1-t)\cdot B},$$

with

$$(t\cdot f_A + (1-t)\cdot f_B)^* = f_{(t\cdot A+(1-t)\cdot B)^{-1}}.$$

The rest of the proof is immediate. □

The following example explains the interest of our approach and the chosen terminology in the above definition.

Example 3.1. Let $E = \mathbb{R}$ and $f(x) = f_a(x) := \frac{1}{2}ax^2,\ g(x) = f_b(x) := \frac{1}{2}bx^2$, with $a > 0, b > 0$. According to (3.1), a simple calculation yields

$$\mathbb{L}(a,b) = \frac{a-b}{\ln a - \ln b} \text{ if } a \neq b,\ \ \mathbb{L}(a,a) = a.$$

That is, $\mathbb{L}(a,b)$ is the known logarithmic mean. Otherwise, Proposition 3.3 gives immediately the arithmetic-logarithmic-harmonic mean inequality

$$\mathbb{H}(a,b) = \frac{2ab}{a+b} \leq \mathbb{L}(a,b) \leq \mathbb{A}(a,b) = \frac{a+b}{2}.$$

We can now present the next definition whose convex integral form appears to be new.

Definition 3.2. The operator $\mathbb{L}(A,B)$, defined by relation (3.1), is the logarithmic operator mean of A and B.

As for all monotone operator means, the explicit formulae of $\mathbb{L}(A,B)$ can be easily deduced from (3.1) and we obtain:

Corollary 3.6. *The logarithmic operator mean of A and B is given by*

$$\mathbb{L}(A,B) = A^{1/2}F\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $F(x) = \frac{x-1}{\ln x}$ for all $x > 0$ with $F(1) = 1$.

According to the above study, it follows that the analogue of the scalar (resp. operator) map $F : x \longmapsto \frac{x-1}{\ln x},\ F(1) = 1$, to convex functionals is $f \longmapsto \mathbb{L}(f,\sigma)$, where $\sigma = \frac{1}{2}\|\cdot\|^2$ is, in the Hilbertian case, the only self-conjugate functional.

4. THE INTERMEDIARY FUNCTIONAL MEANS

In [13], the authors discussed three intermediary functional means constructed from the arithmetic, geometric and harmonic ones. The aim of this section follows the same path.

Let $f, g \in \Gamma_0(E)$, take $f_0 = f,\ g_0 = g$ and consider the following two statements:

(P_1) For all $n \geq 0$, we put $f_{n+1} = \mathbb{L}(f_n, g_n),\ \ g_{n+1} = \mathbb{A}(f_n, g_n)$.
(P_2) For all $n \geq 0$, we put $f_{n+1} = \mathbb{L}(f_n, g_n),\ \ g_{n+1} = \mathbb{H}(f_n, g_n)$.

The fundamental result of this section is the following.

Theorem 4.1. *Let $f, g \in \Gamma_0(E)$ such that $\operatorname{dom} f = \operatorname{dom} g$. Then the sequences (f_n) and (g_n) corresponding to (P_1) (resp. (P_2)) both converge point-wise to the same convex functional. Moreover, denoting these limits by $\mathbb{LA}(f,g)$ and $\mathbb{LH}(f,g)$ respectively, we have the following inequalities*

$$\mathbb{H}(f,g) \leq \mathbb{LH}(f,g) \leq \mathbb{L}(f,g) \leq \mathbb{LA}(f,g) \leq \mathbb{A}(f,g).$$

Proof. We prove the theorem for (P_1), since that of (P_2) can be stated in a similar manner.

First, it is easy to see, by induction, that $f_n \in \Gamma_0(E)$ and $g_n \in \Gamma_0(E)$ for all $n \geq 0$.

By Proposition 3.3, we immediately obtain

$$\forall n \geq 1 \quad f_n \leq g_n,$$

which, with Proposition 3.1, 1. and Proposition 3.2, implies that

$$\forall n \geq 1, \quad f_{n+1} \geq f_n \text{ and } g_{n+1} \leq g_n.$$

Summarizing, we have proved that

$$\mathbb{L}(f,g) = f_1 \leq \cdots \leq f_{n-1} \leq f_n \leq g_n \leq g_{n-1} \leq \cdots \leq g_1 = \mathbb{A}(f,g). \tag{4.1}$$

It follows that, (f_n) is an increasing sequence upper bounded by $g_1 \in \Gamma_0(E)$ and (g_n) is a decreasing one lower bounded by $f_1 \in \Gamma_0(E)$. We deduce that (f_n) and (g_n) both converge point-wise in $\widetilde{\mathbb{R}}^E$. Denoting their convex limits by ϕ and ψ, respectively, we claim that $\phi = \psi$. First, passing to the limit in the inequality $f_n \le g_n$ we obtain $\phi \le \psi$. This, thanks to (4.1), yields

$$\mathbb{L}(f,g) \le \phi \le \psi \le \mathbb{A}(f,g).$$

If $\operatorname{dom} f = \operatorname{dom} g$ then, by (4.1) again, one has $\operatorname{dom}\mathbb{A}(f,g) = \operatorname{dom}\mathbb{L}(f,g)$ which, with the latter inequality, gives $\operatorname{dom}\phi = \operatorname{dom}\psi$.

Now, letting $n \longrightarrow +\infty$ in the relation

$$g_{n+1} = \mathbb{A}(f_n, g_n) := \frac{f_n + g_n}{2},$$

we obtain $2 \cdot \psi = \phi + \psi$, which with $\operatorname{dom}\phi = \operatorname{dom}\psi$ implies that $\phi = \psi$, thus proving the desired result. □

Definition 4.1. The convex functional $\mathbb{LA}(f,g)$ (resp. $\mathbb{LH}(f,g)$) defined by Theorem 4.1 will be called the logarithmic-arithmetic (resp. logarithmic-harmonic) mean of f and g.

In the quadratic case, the above theorem and definition give immediately that of positive operators. In fact, let A and B be two positive (invertible) linear operators from E into E^*, take $A_0 = A,\ B_0 = B$ and define the two quadratic processes:

(QP_1) For all $n \ge 0$, we put $A_{n+1} = \mathbb{L}(A_n, B_n), \quad B_{n+1} = \mathbb{A}(A_n, B_n)$.
(QP_2) For all $n \ge 0$, we put $A_{n+1} = \mathbb{L}(A_n, B_n), \quad B_{n+1} = \mathbb{H}(A_n, B_n)$.

From the above, we obtain the following quadratic version.

Corollary 4.2. *Let A and B as in the above. Then the sequences (A_n) and (B_n) corresponding to (QP_1) (resp. (QP_2)) both converge strongly to the same positive operator $\mathbb{LA}(A,B)$ (resp.$\mathbb{LH}(A,B)$) satisfying*

$$\mathbb{H}(A,B) \le \mathbb{LH}(A,B) \le \mathbb{L}(A,B) \le \mathbb{LA}(A,B) \le \mathbb{A}(A,B).$$

Similar to the functional case, we have the following definition.

Definition 4.2. The above positive operator $\mathbb{LA}(A,B)$ (resp. $\mathbb{LH}(A,B)$) will be called the logarithmic-arithmetic (resp. logarithmic-harmonic) mean of A and B.

Remark 1. Let $f, g \in \Gamma_0(E)$ and define the map

$$T(f,g) = \left(\mathbb{L}(f,g), \mathbb{A}(f,g)\right), \quad \text{resp. } T(f,g) = \left(\mathbb{L}(f,g), \mathbb{H}(f,g)\right).$$

If T^n denotes the $n-$th iterate of T, Theorem 4.1 tells us that there exists a convex functional $\mathbb{F}$ such that

$$\lim_{n\uparrow+\infty} T^n(f,g) = (\mathbb{F}, \mathbb{F}).$$

Analogous deductions can be made for Corollary 4.2.

5. LOGARITHMIC MEAN OF SEVERAL VARIABLES

Below, we outline the procedure to extend the above logarithmic mean from two functional variables to three or more.

Let $m \ge 2$ be an integer and define

$$\Delta_{m-1} = \left\{ (t_1, t_2, \dots, t_{m-1}) \in \mathbb{R}^{m-1}, \ \sum_{i=1}^{m-1} t_i \le 1, \ \ t_i \ge 0 \ \text{ for } \ 1 \le i \le m-1 \right\}.$$

Let us put

$$t_m = 1 - \sum_{i=1}^{m-1} t_i,$$

and by analogy to Definition 3.1, we find a real $a_m > 0$ such that the expression

$$\left(a_m \int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i f_i \right)^* dt_1 dt_2 \dots dt_{m-1} \right)^*,$$

is a reasonable extension of $\mathbb{L}(f,g)$. For this, we compute a_m by requiring that (analogously with $\mathbb{L}(f,f) = f$)

$$a_m \cdot \int_{\Delta_{m-1}} dt_1 dt_2 \dots dt_{m-1} = 1.$$

A classical integration gives

$$a_m^{-1} = \int_{\Delta_{m-1}} dt_1 dt_2 \dots dt_{m-1} = \frac{1}{(m-1)!}.$$

Now, we can introduce the following definition.

Definition 5.1. Let $f_1, f_2, \dots, f_m \in \Gamma_0(E)$ such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \emptyset$. The logarithmic functional mean of $f_1, f_2, \dots, f_m$ is given by

$$\mathbb{L}(f_1, f_2, \dots, f_m) = \left((m-1)! \int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i f_i \right)^* dt_1 dt_2 \dots dt_{m-1} \right)^*,$$

where $t_m = 1 - \sum_{i=1}^{m-1} t_i$.

From the above definition, the properties of $\mathbb{L}(f_1, f_2, \dots, f_m)$ can be stated, in a similar manner to that of the two functional variables case. In the following, we summarize these properties whose proofs are omitted (and we leave it to the reader, since they are analogous to that of $\mathbb{L}(f,g)$). To simplify the notations, we write $\mathbb{L}(F)$ instead of $\mathbb{L}(f_1, f_2, \dots, f_m)$ with $F = (f_1, f_2, \dots, f_m) \in (\Gamma_0(E))^m$, and $F_\tau = (f_{\tau(1)}, f_{\tau(2)}, \dots, f_{\tau(m)})$ for a permutation τ of $\{1, 2, \dots, m\}$. With this, we define, for $\lambda > 0$

$$\lambda \cdot F = (\lambda \cdot f_1, \lambda \cdot f_2, \dots, \lambda \cdot f_m) \quad \text{and} \quad F \cdot \lambda = (f_1 \cdot \lambda, f_2 \cdot \lambda, \dots, f_m \cdot \lambda).$$

If $F = (f, f, \dots, f)$ with $f \in \Gamma_0(E)$ we write $\mathbb{L}(f)$.

Proposition 5.1. *With the above, the following statements hold true:*

(1) $\mathbb{L}(F) \in \Gamma_0(E)$, $\mathbb{L}(f) = f$, *and* $\mathbb{L}(F) = \mathbb{L}(F_\tau)$ *for all permutations* τ *of the set* $\{1, 2, \dots, m\}$.
(2) *For all* $\lambda > 0$, $\mathbb{L}(\lambda \cdot F) = \lambda \cdot \mathbb{L}(F)$ *and* $\mathbb{L}(F \cdot \lambda) = \mathbb{L}(F) \cdot \lambda$.
(3) *If* $f_i, g_i \in \Gamma_0(E)$ *such that* $f_i \leq g_i$ *for all* $i = 1, 2, \dots, m$, *then* $\mathbb{L}(F) \leq \mathbb{L}(G)$ *with* $G = (g_1, g_2, \dots, g_m)$.

Now, for $F = (f_1, f_2, \dots, f_m) \in (\Gamma_0(E))^m$ we put

$$\mathbb{A}(F) = \sum_{i=1}^{m} \frac{f_i}{m}, \qquad \mathbb{H}(F) = \left(\sum_{i=1}^{m} \frac{f_i^*}{m} \right)^*,$$

which are, respectively, the arithmetic and harmonic means of $f_1, f_2, \dots, f_m$.

Proposition 5.2. *With the above conditions, the following arithmetic-logarithmic-harmonic functional mean inequality holds*

$$\mathbb{H}(F) \leq \mathbb{L}(F) \leq \mathbb{A}(F).$$

Proof. Firstly, for brevity, we only present the fundamental points of this proof. By virtue of the relation

$$(m-1)!\int_{\Delta_{m-1}} dt_1 dt_2 \dots dt_m = 1,$$

Lemma 2.1, with Definition 5.1, implies that

$$\mathbb{L}(F) \le (m-1)!\int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i \cdot f_i\right) dt_1 dt_2 \dots dt_{m-1},$$

with $t_m = 1 - \sum_{i=1}^{m-1} t_i$. By the symmetry of Δ_m, a classical computation yields

$$\int_{\Delta_{m-1}} t_i dt_1 dt_2 \dots dt_{m-1} = \frac{1}{m!},$$

for all $i = 1, 2, \dots, m$. Substituting this in the above, we obtain the arithmetic-logarithmic inequality. The logarithmic-harmonic one can be similarly obtained to that of Proposition 3.3, which completes the proof. □

The main interest of our functional approach appears in its convex form with the simple related proofs. In what follows, we provide another example explaining this situation.

Proposition 5.3. *Let $F = (f_i)_{i=1}^m$ with $f_i = f_{A_i}$, where $A_i : E \longrightarrow E^*$ are positive invertible operators. If we set $A = (A_1, A_2, \dots, A_m)$, then*

$$\mathbb{L}(F) = f_{\mathbb{L}(A)},$$

where

$$\mathbb{L}(A) = \left((m-1)!\int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i A_i\right)^{-1} dt_1 dt_2 \dots dt_{m-1}\right)^{-1}. \tag{5.1}$$

$\mathbb{L}(A)$ is called the logarithmic operator mean of $A_1, A_2, \dots, A_m$.

Proof. It is a simple exercise for the reader. □

Corollary 5.4. *For all positive invertible operators $A_1, A_2, \dots, A_m$, we have the arithmetic-logarithmic-harmonic operator mean inequality*

$$\mathbb{H}(A) \le \mathbb{L}(A) \le \mathbb{A}(A),$$

where

$$\mathbb{A}(A) = \sum_{i=1}^{m} \frac{A_i}{m}, \qquad \mathbb{H}(A) = \left(\sum_{i=1}^{m} \frac{A_i^{-1}}{m}\right)^{-1},$$

and $\mathbb{L}(A)$ is defined by (5.1).

Proof. It is sufficient to combine Proposition 5.2 with Proposition 5.3. □

Example 5.1. Let a, b, c be three positive reals. According to (5.1), we wish to compute the logarithmic mean of a, b and c. First, it is easy to see that $\mathbb{L}(a, a, a) = a$ for all $a > 0$. For $a \ne b, a \ne c$ and $b \ne c$, a classical integration yields

$$\mathbb{L}(a, b, c) = \frac{1}{2}\frac{(a-b)(b-c)(c-a)}{a(c-b)\ln a + b(a-c)\ln b + c(b-a)\ln c},$$

see also [12]. By symmetry (Proposition 5.1,1.), for $a \ne c$ one has

$$\mathbb{L}(a, a, c) = \mathbb{L}(a, c, a) = \mathbb{L}(c, a, a),$$

which naturally satisfies

$$\mathbb{L}(a,a,c)=\lim_{b\longrightarrow a}\mathbb{L}(a,b,c).$$

After a simple computation of this limit (or using (5.1)), we obtain (for $a\neq c$)

$$\mathbb{L}(a,a,c)=\frac{(a-c)^2}{2\,(a-c+c(\ln c-\ln a))}.$$

The arithmetic-logarithmic-harmonic mean inequality is immediately given by

$$\frac{3\cdot abc}{ab+bc+ca}\leq\mathbb{L}(a,b,c)\leq\frac{a+b+c}{3}.$$

These latter inequalities are not directly immediate, that proves the interest of our approach.

Finally, we end this section by introducing another functional mean of three variables constructed from the arithmetic, logarithmic and harmonic ones.

Let $f,g,h\in\Gamma_0(E)$, take $f_0=f,g_0=g,h_0=h$ and define the recursive process

$$\forall n\geq 0,\quad f_{n+1}=\mathbb{H}(f_n,g_n,h_n),\quad g_{n+1}=\mathbb{L}(f_n,g_n,h_n),\quad h_{n+1}=\mathbb{A}(f_n,g_n,h_n).$$

Clearly, f_n,g_n and h_n belong to $\Gamma_0(E)$ for each $n\geq 0$.

Theorem 5.5. *Let $f,g,h\in\Gamma_0(E)$ such that $\operatorname{dom} f=\operatorname{dom} g=\operatorname{dom} h$. Then the sequences $(f_n),(g_n)$ and (h_n) both converge point-wise to the same limit $\mathbb{ALH}(f,g,h)$ satisfying*

$$\mathbb{H}(f,g,h)\leq\mathbb{ALH}(f,g,h)\leq\mathbb{A}(f,g,h).$$

Proof. By Proposition 5.2, we obtain

$$\forall n\geq 1\qquad f_n\leq g_n\leq h_n,$$

which, with Proposition 5.1,1. and 3., implies that

$$\forall n\geq 1,\qquad f_{n+1}\geq f_n\ \text{ and }\ h_{n+1}\leq h_n.$$

In summary, we have shown that

$$\mathbb{H}(f,g,h)=f_1\leq\cdots\leq f_{n-1}\leq f_n\leq g_n\leq h_n\leq h_{n-1}\leq\cdots\leq h_1=\mathbb{A}(f,g,h). \tag{5.2}$$

We conclude that (f_n) is increasing upper bounded by h_1 and (h_n) is decreasing lower bounded by f_1. Thus they converge point-wise in $\widetilde{\mathbb{R}}^E$ whose limits are denoted respectively by ϕ and ψ. Otherwise, from the relation

$$3\cdot h_{n+1}=f_n+g_n+h_n, \tag{5.3}$$

we deduce that (g_n) is also point-wise convergent to a limit θ. Now, letting $n\longrightarrow+\infty$ in relation (5.2), we can write

$$\mathbb{H}(f,g,h)\leq\phi\leq\theta\leq\psi\leq\mathbb{A}(f,g,h). \tag{5.4}$$

Since $\operatorname{dom} f=\operatorname{dom} g=\operatorname{dom} h$ then, following [15], one has $\operatorname{dom}\mathbb{H}(f,g,h)=\operatorname{dom}\mathbb{A}(f,g,h)=\operatorname{dom} f$, and by (5.4), we obtain

$$\operatorname{dom}\phi=\operatorname{dom}\theta=\operatorname{dom}\psi. \tag{5.5}$$

Passing to the limit in (5.3), we have $3\cdot\psi=\phi+\theta+\psi$ and so, with (5.5), $2\cdot\psi=\phi+\theta$. This, when combined with (5.4), yields $\phi=\psi=\theta$. This completes the proof. $\square$

As in the above section, if the convex functionals f, g and h are quadratic then we immediately obtain a similar result for positive operators. Indeed, let $A, B, C : E \longrightarrow E^*$ be three positive (invertible) operators and let $A_0 = A, B_0 = B, C_0 = C$ and define the quadratic sequences

$$\forall n \geq 0, \quad A_{n+1} = \mathbb{H}(A_n, B_n, C_n), \ \ B_{n+1} = \mathbb{L}(A_n, B_n, C_n), \ \ C_{n+1} = \mathbb{A}(A_n, B_n, C_n).$$

Corollary 5.6. *Let A, B, C as in the above. The sequences $(A_n), (B_n)$ and (C_n) both converge strongly to the same positive operator $\mathbb{ALH}(A, B, C)$, called the arithmetic-logarithmic-harmonic operator mean, which satisfies*

$$\mathbb{H}(A, B, C) \leq \mathbb{ALH}(A, B, C) \leq \mathbb{A}(A, B, C).$$

Definition 5.2. $\mathbb{ALH}(f, g, h)$ defined by Theorem 5.5 (resp. $\mathbb{ALH}(A, B, C)$ defined by Corollary 5.6) will be called the arithmetic-logarithmic-harmonic functional mean of f, g and h (resp. operator mean of A, B and C).

As in Remark 1, Theorem 5.5 (and analogously Corollary 5.6) can be summarized by saying that there exists a convex functional $\mathbb{F}$ such that

$$\lim_{n\uparrow+\infty} T^n(f, g, h) = (\mathbb{F}, \mathbb{F}, \mathbb{F}),$$

where

$$T(f, g, h) = (\mathbb{A}(f, g, h), \mathbb{L}(f, g, h), \mathbb{H}(f, g, h)).$$

REFERENCES

[1] M. ATTEIA AND M. RAÏSSOULI, Self dual operator on convex functionals, geometric mean and square root of convex functionals, *Journal of Convex Analysis*, **8**(1) (2001), 223–240.

[2] J.P. AUBIN, *Analyse non Linéaire et ses Motivations Economiques*, Masson, 1983.

[3] J.S. AUJLA AND M.S. RAWLA, Some results on operator means and shorted operators, *Rend. Sem. Mat. Univ. Pol. Torino*, **59**(3) (2001), 189–198.

[4] H. BREZIS, *Analyse Fonctionnelle, Théorie et Applications*, Masson, 1983.

[5] A. EL BIARI, R. ELLAIA AND M. RAÏSSOULI, Stability of geometric and harmonic functional means, *Journal of Convex Analysis*, **10**(1) (2003), 199–210.

[6] J.I. FUJII, Arithmetico-geometric mean of operators, *Math. Japonica*, **23** (1978), 667–669.

[7] J.I. FUJII, On geometric and harmonic means of positive operators, *Math. Japonica*, **24**(2) (1979), 203–207.

[8] J.I. FUJII, Kubo-Ando theory of convex functional means, *Scientiae Mathematicae Japonicae*, **7** (2002), 299–311.

[9] F. KUBO AND T. ANDO, Means of positive linear operators, *Math. Ann.*, **246** (1980), 205–224.

[10] P.J. LAURENT, *Approximation et Optimisation*, Hermann, 1972.

[11] R.D. NUSSBAUM AND J.E. COHEN, The arithmetico-geometric means and its generalizations for non commuting linear operators, *Ann. Sci. Norm. Sup. Sci.*, **15** (1989), 239–308.

[12] A.O. PITTENGER, The logarithm mean in n variables, *Amer. Math. Monthly*, **92** (1985), 99–104.

[13] M. RAÏSSOULI AND M. CHERGUI, Arithmetico-geometric and geometrico-harmonic means of two convex functionals, *Scientiae Mathematicae Japonicae*, **55**(3) (2002), 485–492.

[14] M. RAÏSSOULI AND H. BOUZIANE, Functional logarithm in the sense of convex analysis, *J. of Convex Analysis*, **10**(1) (2003), 229–244.

[15] M. RAÏSSOULI AND H. BOUZIANE, Functional limited development in convex analysis, *Annales des Sciences Mathématiques du Québec*, **29**(1) (2005), 97–110.

[16] M. RAÏSSOULI AND H. BOUZIANE, Arithmetico-geometrico-harmonic functional mean in the sense of convex analysis, *Annales des Sciences Mathématiques du Québec*, **30**(1) (2006), 79–107.

[17] M. RAÏSSOULI, Relative functional entropy in convex analysis, *Scientiae Mathematicae Japonicae*, 2008, submitted.

[18] M. RAÏSSOULI, Tsallis relative entropy for convex functionals, *International Journal of Pure and Applied Mathematics*, **51**(4) (2009), 555–563.

[19] M. RAÏSSOULI, Some product operations involving convex functionals, in preparation.