INEQUALITIES THAT LEAD TO EXPONENTIAL STABILITY AND INSTABILITY IN DELAY DIFFERENCE EQUATIONS

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Abstract: We use Lyapunov functionals to obtain sufficient conditions that guarantee exponential stability of the zero solution of the delay difference equation

$$x(t+1) = a(t)x(t) + b(t)x(t-h).$$

The highlight of the paper is the relaxing of the condition $|a(t)| < 1$. An instability criteria for the zero solution is obtained. Moreover, we will provide an example, in which we show that our theorems provide an improvement of some of the recent literature.

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1. Introduction

In this paper we consider the scalar linear difference equation with multiple delays

$$x(t+1) = a(t)x(t) + b(t)x(t-h), \tag{1.1}$$

where $h \in \mathbb{Z}^+$ and $a, b : \mathbb{Z}^+ \to \mathbb{R}$. In this paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}^+$ denotes the set of positive integers. We will use Lyapunov functionals and obtain some inequalities regarding the solutions of (1.1) from which we can deduce the exponential asymptotic stability of the zero solution. Also, we will provide a criteria for the instability of the zero solution of (1.1) by means of a Lyapunov functional.

Due to the choice of the Lyapunov functionals, we will deduce some inequalities on all solutions. As a consequence, the exponential stability of the zero solution is concluded. Consider the kth-order scalar difference equation

$$x(t+k) + p_1 x(t+k-1) + p_2(t+k-2) + \cdots + p_k(t) = 0, \tag{1.2}$$

where the p_i's are real numbers. It is well known that the zero solution of (1.2) is asymptotically stable if and only if $|\lambda| < 1$ for every characteristic root λ of (1.2). There is no such criteria to test for exponential stability of the zero solution of equations that are similar to (1.2). This itself highlights the importance of constructing a suitable Lyapunov function that leads to exponential stability. This paper is devoted to constructing a Lyapunov functional that yields exponential stability. In such instances, one faces the tedious task of relating the Lyapunov functional back to the solution x so that stability can be deduced. In the paper [2], the author obtained easily verifiable conditions that guaranteed the exponential stability of an equation similar to (1.1). Later on, the authors of [1] used a recursive approach method and improved the results of [2]. We are happy to say that we will furnish an example that will improve both the results of [2] and [1]. There are not many papers dealing with the study of exponential stability of delay difference equations and this paper intends

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to do so, in addition to improving recent results in which Lyapunov's method was not used.

Let $\psi : [-h, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-h \le s \le 0} |\psi(s)|.$$

It should cause no confusion to denote the norm of a function $\varphi : [-h, \infty) \to (-\infty, \infty)$ with

$$||\varphi|| = \sup_{-h \le s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t + \tau), \tau \in [-h, 0]$ as long as $x(t + \tau)$ is defined. Thus, x_t is a function mapping an interval $[-h, 0]$ into $\mathbb{R}$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \ge t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-h, 0]$.

In preparation for our main results, we note that (1.1) is equivalent to

$$\triangle x(t) = (a(t) + b(t + h) - 1)\, x(t) - \triangle_t \sum_{s=t-h}^{t-1} b(s + h)x(s). \tag{1.3}$$

We end this section with the following definition.

Definition 1.1. *The zero solution of* (1.1) *is said to be exponentially stable if any solution* $x(t, t_0, \psi)$ *of* (1.1) *satisfies*

$$|x(t, t_0, \psi)| \le C\,(||\psi||, t_0)\, \zeta^{\gamma(t-t_0)}, \quad \textit{for all } t \ge t_0,$$

where ζ *is constant with* $0 < \zeta < 1$, $C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, *and* γ *is a positive constant. The zero solution of* (1.1) *is said to be uniformly exponentially stable if* C *is independent of* t_0.

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2. Exponential Stability

Now we turn our attention to the exponential decay of solutions of equation (1.1). For simplicity, we let

$$Q(t) = b(t+h) + a(t) - 1.$$

Lemma 2.1. *Assume that for* $\delta > 0$,

$$-\frac{\delta}{(\delta+1)h} \leq Q(t) \leq -\delta h b^2(t+h) - Q^2(t), \tag{2.1}$$

holds. If

$$V(t) = \left[x(t) + \sum_{s=t-h}^{t-1} b(s+h)x(s)\right]^2 + \delta \sum_{s=-h}^{-1} \sum_{z=t+s}^{t-1} b^2(z+h)x^2(z), \tag{2.2}$$

then, based on the solutions of (1.1) we have

$$\triangle V(t) \leq Q(t)V(t). \tag{2.3}$$

Proof. First we note that due to condition (2.1), $Q(t) < 0$ for all $t \geq 0$. Also, we use that fact that if $u(t)$ is a sequence, then $\triangle u^2(t) = u(t+1)\triangle u(t) + u(t)\triangle u(t)$. For more on the calculus of difference equations we refer the reader to [3] and [4]. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (2.2). Then based on the solutions of (1.2) we have

$$\triangle V(t) = \left[x(t+1) + \sum_{s=t-h+1}^{t} b(s+h)\right] \triangle \left[x(t) + \sum_{s=t-h}^{t-1} b(s+h)x(s)\right] \tag{2.4}$$

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$$
\begin{aligned}
&+\left[x(t)+\sum_{s=t-h}^{t-1} b(s+h)\right] \triangle \left[x(t)+\sum_{s=t-h}^{t-1} b(s+h)x(s)\right] \\
&+\delta \sum_{s=-h}^{-1}\left(b^2(t+h)x^2(t)-b^2(t+s+h)x^2(t+s)\right) \\
= &\left[(Q(t)+1)x(t)+\sum_{s=t-h}^{t-1} b(s+h)x(s)\right] Q(t)x(t) \\
&+\left[x(t)+\sum_{s=t-h}^{t-1} b(s+h)x(s)\right] Q(t)x(t) \\
&+\delta h b^2(t+h)x^2(t)-\delta \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \\
= &\, Q(t)x^2(t)+2Q(t)x(t)\sum_{s=t-h}^{t-1} b(s+h)x(s) \\
&+\left(Q^2(t)+Q(t)+\delta h b^2(t+h)\right)x^2(t)-\delta \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \\
= &\, Q(t)V(t)+\left(Q^2(t)+Q(t)+\delta h b^2(t+h)\right)x^2(t) \\
&-\delta Q(t)\sum_{s=-h}^{-1}\sum_{z=t+s}^{t-1} b^2(z+h)x^2(z) \\
&-\delta \sum_{s=-h}^{-1} b^2(t+s+h)x^2(t+s)-Q(t)\left(\sum_{s=t-h}^{t-1} b(s+h)x(s)\right)^2.
\end{aligned}
$$

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In what follows we perform some calculations to simplify (2.4). First, if we let $u = t + s$, then

$$-\delta \sum_{s=-h}^{-1} b^2(t+s+h)x^2(t+s) = -\delta \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s). \tag{2.5}$$

Also, with the aid of Hölder's inequality, we have

$$\left(\sum_{s=t-h}^{t-1} b(s+h)x(s) \right)^2 \le h \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s). \tag{2.6}$$

Invoking (2.1) and substituting expressions (2.5) – (2.6) into (2.4), yields

$$\begin{aligned} \triangle V(t) &\le Q(t)V(t) + \left(Q^2(t) + Q(t) + \delta h b^2(t+h) \right) x^2(t) \\ &\qquad + \left[-(\delta+1) hQ(t) - \delta \right] \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \\ &\le Q(t)V(t). \end{aligned} \tag{2.7}$$

□

Theorem 2.2. *Let the hypothesis of Lemma 2.4 hold. Suppose there exists a number $\alpha < 1$ such that*

$$0 < b(t+s) + a(t) \le \alpha.$$

Then any solution $x(t) = x(t, t_0, \psi)$ of (1.1) satisfies the exponential inequalities

$$|x(t)| \le \sqrt{\frac{h+\delta}{\delta} V(t_0) \prod_{s=t_0}^{t-1} (b(s+h) + a(s))} \tag{2.8}$$

for $t \ge t_0$.

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Proof. First we note that condition (2.1) implies that there exists some positive number $\alpha < 1$ such that $|b(t+s)+a(t)| < 1$. Now by changing the order of summation we have

$$\begin{aligned}\delta \sum_{s=-h}^{-1} \sum_{z=t+s}^{t-1} b^2(z+h)x^2(z) &= \delta \sum_{z=t-h}^{t-1} \sum_{s=-h}^{z-1} b^2(z+h)x^2(z) \\ &= \delta \sum_{z=t-h}^{t-1} b^2(z+h)x^2(z)(z-t+h+1) \\ &\geq \delta \sum_{z=t-h}^{t-1} b^2(z+h)x^2(z),\end{aligned}$$

where we have used the fact that

$$t-h \leq z \leq t-1 \implies 1 \leq z-t+h+1 \leq h.$$

Let $V(t)$ be given by (2.2).

$$\left(\sum_{z=t-h}^{t-1} b(z+h)x(s)\right)^2 \leq h \sum_{z=t-h}^{t-1} b^2(z+h)x^2(s).$$

Hence,

$$\delta \sum_{s=-h}^{-1} \sum_{z=t+s}^{t-1} b^2(z+h)x^2(z) \geq \frac{\delta}{h}\left(\sum_{z=t-h}^{t-1} b(z+h)x(s)\right)^2.$$

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Let $V(t)$ be given by (2.2). Then

$$V(t) = \left[x(t) + \sum_{s=t-h}^{t-1} b(s+h)x(s)\right]^2 + \delta \sum_{s=-h}^{-1} \sum_{z=t+s}^{t-1} b^2(z+h)x^2(z)$$

$$\geq \left[x(t) + \sum_{s=t-h}^{t-1} b(s+h)x(s)\right]^2 + \frac{\delta}{h}\left(\sum_{z=t-h}^{t-1} b(z+h)x(s)\right)^2$$

$$+ \frac{\delta}{h+\delta}x^2(t) + \left[\sqrt{\frac{h}{h+\delta}}x(t) + \sqrt{\frac{h+\delta}{h}} \sum_{z=t-h}^{t-1} b(z+h)x(s)\right]^2$$

$$\geq \frac{\delta}{h+\delta}x^2(t).$$

Consequently,

$$\frac{\delta}{h+\delta}x^2(t) \leq V(t).$$

From (2.7) we get

$$V(t) \leq V(t_0) \prod_{s=t_0}^{t-1} \left(b(s+h) + a(s)\right).$$

The results follow from the inequality

$$\frac{\delta}{h+\delta}x^2(t) \leq V(t).$$

This completes the proof. □

Corollary 2.3. *Assume that the hypotheses of Theorem 2.2 hold. Then the zero solution of (1.1) is exponentially stable.*

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Proof. From inequality (2.8) we have that

$$|x(t)| \le \sqrt{\frac{h+\delta}{\delta} V(t_0) \prod_{s=t_0}^{t-1} (b(s+h)+a(s))}$$
$$\le \sqrt{\frac{h+\delta}{\delta} V(t_0)\alpha^{t-t_0}}$$

for $t \ge t_0$. The proof is completed since $\alpha \in (0,1)$. □

Next we give a simple example to show that condition (2.1) can be easily verified and moreover, we take $|a(t)| > 1$.

Example 2.1. Let $a = 1.2, b = -0.3, h = 1,$ and $\delta = 0.5$. Then one can easily verify that (2.1) is satisfied. Hence the zero solution of the delay difference equation

$$x(t+1) = 1.2x(t) - 0.3x(t-1)$$

is exponentially stable.

It is worth mentioning that in both papers [5] and [6] it was assumed that

$$\prod_{s=0}^{t-1} a(s) \to 0, \text{ as } t \to \infty$$

for the asymptotic stability. Of course our $a = 1.2$ does not satisfy such a condition, and yet we concluded exponential stability. Also, to compare our results with the results obtained in [2] we state the following.

Lemma 2.4 ([2]). *If there exists $\lambda \in (0,1)$ such that*

$$\left|\prod_{j=0}^{N} a(n-j) + b(n)\right| + \sum_{s=1}^{N} \left|\prod_{j=0}^{s-1} a(n-j)\right| |b(n-s)| \le \lambda, \tag{2.9}$$

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for large n, then the zero solution of

$$x(n+1) = a(n)x(n) + b(n)x(n-N)$$

is globally exponentially stable.

It can be easily seen that condition (2.9) cannot be satisfied for the data given in the above example. Next we state two major results from [2] so that we can compare them with our example.

Lemma 2.5 ([1]). *Let $0 < \gamma < 1$ and $\sum_{l=2}^{m} |a_l(n)| + |1 - a_1(n)| \leq \gamma$ for n large enough. Then the equation*

$$x(n+1) - x(n) = -a_1(n)x(n) - \sum_{l=2}^{m} a_l(n)x(g_l(n)) \tag{2.10}$$

is exponentially stable. Here $n - T \leq g_l(n) \leq n$ for some integer $T > 0$.

Lemma 2.6 ([1]). *Suppose that for some $\gamma \in (0,1)$ the following inequality is satisfied for n large enough:*

$$\sum_{k=2}^{m} |a_k(n)| \sum_{j=g_k(n)}^{n-1} \sum_{l=1}^{m} |a_l(j)| + \left| 1 - \sum_{k=1}^{m} a_k(n) \right| \leq \gamma. \tag{2.11}$$

Then (2.10) is exponentially stable.

In the spirit of (2.10) we rewrite the difference equation in Example 2.1 as

$$x(t+1) - x(n) = 0.2x(t) - 0.3x(t-1).$$

Then the condition in Lemma 2.6 is equivalent to

$$|a_2(n)| + |1 - a_1(n)| = 0.3 + |1 + 0.2| > 1.$$

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Also, condition (2.11) is equivalent to

$$|a_2(n)||a_1(n)| + |1 - a_1(n)| = 0.3(0.2) + |1 + 0.2| > 1.$$

Thus, we have demonstrated that our results improve the results of [1] and [2].

We end this paper by giving a criteria for instability via Lyapunov functionals.

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3. Criteria For Instability

In this section, we use a non-negative definite Lyapunov functional and obtain criteria that can be easily applied to test for the instability of the zero solution of (1.1).

Theorem 3.1. *Let $H > h$ be a constant. Assume that $Q(t) > 0$ such that*

$$Q^2(t) + Q(t) - Hb^2(t+h) \geq 0. \tag{3.1}$$

If

$$V(t) = \left[x(t) + \sum_{s=t-h}^{t-1} b(s+h)x(s)\right]^2 - H \sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \tag{3.2}$$

then, based on the solutions of (1.1) we have

$$\triangle V(t) \geq Q(t)V(t).$$

Proof. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (3.2). Since the calculation is similar to the one in Lemma 2.4, we arrive at

$$\begin{aligned} \triangle V(t) &= Q(t)V(t) + \left(Q^2(t) + Q(t) - Hb^2(t+h)\right) x^2(t) \\ &\quad + Hb^2(t)x^2(t-h) - Q(t)\left(\sum_{s=t-h}^{t-1} b(s+h)x(s)\right)^2 \\ &\quad + HQ(t)\sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \\ &\geq Q(t)V(t), \end{aligned} \tag{3.3}$$

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where we have used

$$\left(\sum_{s=t-h}^{t-1} b(s+h)x(s)\right)^2 \le h\sum_{s=t-h}^{t-1} b^2(s+h)x^2(s) \le H\sum_{s=t-h}^{t-1} b^2(s+h)x^2(s)$$

and (3.1). This completes the proof. □

We remark that condition (3.1) is satisfied for

$$Q(t) \ge \frac{-1+\sqrt{1+4Hb^2(t+h)}}{2}.$$

Theorem 3.2. *Suppose hypotheses of Theorem 3.1 hold. Then the zero solution of (1.1) is unstable, provided that*

$$\prod^{\infty}(b(s+h)+a(s)) = \infty.$$

Proof. From (3.3) we have

$$V(t) \ge V(t_0)\prod_{s=t_0}^{t-1}(b(s+h)+a(s)). \tag{3.4}$$

Let $V(t)$ be given by (3.2). Then

$$V(t) = x^2(t) + 2x(t)\sum_{t-h}^{t-1} b(s+h_i)x(s) + \left[\sum_{t-h}^{t-1} b(s+h_i)x(s)\right]^2 - H\sum_{t-h}^{t-1} b^2(s+h)x^2(s). \tag{3.5}$$

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Let $\beta = H - h$. Then from

$$\left(\frac{\sqrt{h}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{h}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{h}{\beta}a^2 + \frac{\beta}{h}b^2.$$

With this in mind we arrive at,

$$\begin{aligned}
2x(t)\sum_{t-h}^{t-1} b(s+h)x(s) &\leq 2|x(t)|\left|\sum_{t-h}^{t-1} b(s+h)x(s)\right| \\
&\leq \frac{h}{\beta}x^2(t) + \frac{\beta}{h}\left[\sum_{t-h}^{t-1} b(s+h)x(s)\right]^2 \\
&\leq \frac{h}{\beta}x^2(t) + \frac{\beta}{h}h\sum_{t-h}^{t-1} b^2(s+h)x^2(s).
\end{aligned}$$

A substitution of the above inequality into (3.5) yields,

$$\begin{aligned}
V(t) &\leq x^2(t) + \frac{h^*}{\beta}x^2(t) + (\beta + h^* - H)\sum_{t-h}^{t-1} b^2(s+h)x^2(s) \\
&= \frac{\beta + h}{\beta}x^2(t) \\
&= \frac{H}{H-h}x^2(t).
\end{aligned}$$

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Using inequality (3.4), we get

$$|x(t)| \geq \sqrt{\frac{H}{H-h^*} D}\, V^{1/2}(t)$$

$$= \sqrt{\frac{H}{H-h}}\, V^{1/2}(t_0) V(t_0) \left(\prod_{s=t_0}^{t-1} [b(s+h) + a(s)] \right)^{\frac{1}{2}} .$$

This completes the proof. According to Theorem 3.2, the zero solution of

$$x(t+1) = 0.9x(t) + 0.2x(t-1)$$

is unstable when $H \geq 1.1$. □

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