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SOME RESULTS ON A GENERALIZED USEFUL INFORMATION MEASURE

¹ABUL BASAR KHAN, ¹BILAL AHMAD BHAT AND ²S. PIRZADA

¹Division of Agricultural Economics and Statistics, Sher-e-Kashmir
University of Agricultural Sciences and Technology Jammu
Faculty of Agriculture
Main Campus Chatha-180009 India.
EMail: bhat_bilal@rediffmail.com

²Department of Mathematics
University of Kashmir
Srinagar-190006, India
EMail: sdpirzada@yahoo.co.in



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Abstract

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Abstract

A parametric mean length is defined as the quantity

$${}_{\alpha\beta}L_u = \frac{\alpha}{\alpha-1} \left[1 - \sum P_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-n_i(\frac{\alpha-1}{\alpha})} \right],$$

where $\alpha \neq 1, \sum p_i = 1$

this being the useful mean length of code words weighted by utilities, u_i . Lower and upper bounds for ${}_{\alpha\beta}L_u$ are derived in terms of useful information for the incomplete power distribution, p^β .

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1. Introduction

Consider the following model for a random experiment S ,

$$S_N = [E; P; U]$$

where $E = (E_1, E_2, \dots, E_n)$ is a finite system of events happening with respective probabilities $P = (p_1, p_2, \dots, p_N)$, $p_i \geq 0$, and $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, \dots, u_N)$, $u_i > 0$, $i = 1, 2, \dots, N$. Denote the model by E , where

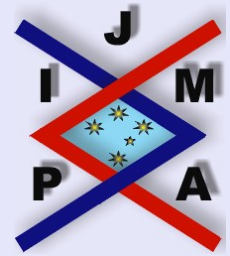
$$(1.1) \quad E = \begin{bmatrix} E_1 & E_2 & \cdots & E_N \\ p_1 & p_2 & \cdots & p_N \\ u_1 & u_2 & \cdots & u_N \end{bmatrix}$$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [3] proposed a measure of information called ‘useful information’ for this scheme, given by

$$(1.2) \quad H(U; P) = - \sum u_i p_i \log p_i,$$

where $H(U; P)$ reduces to Shannon’s [8] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each i . Throughout the paper, \sum will stand for $\sum_{i=1}^N$ unless otherwise stated and logarithms are taken to base D ($D > 1$).

Guiasu and Picard [5] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords w_1, w_2, \dots, w_N having lengths



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n_1, n_2, \dots, n_N and satisfying Kraft's inequality [4]

$$(1.3) \quad \sum_{i=1}^N D^{-n_i} \leq 1,$$

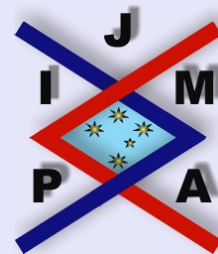
where D is the size of the code alphabet. The useful mean length L_u of code was defined as

$$(1.4) \quad L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i}$$

and the authors obtained bounds for it in terms of $H(U; P)$.

Longo [8], Gurdial and Pessoa [6], Khan and Autar [7], Autar and Khan [2] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under condition (1.3) of unique decipherability.

In this paper, we study some coding theorems by considering a new function depending on the parameters α and β and a utility function. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature (see C. Arndt [1]). The function under study is closely related to Tsallis entropy which is used in physics.



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2. Coding Theorems

Consider a function

$$(2.1) \quad {}_{\alpha\beta}H(U; P) = \frac{\alpha}{\alpha - 1} \left[1 - \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} \right],$$

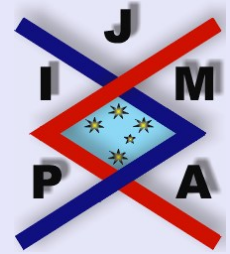
where $\alpha > 0$ ($\neq 1$), $\beta > 0$, $p_i \geq 0$, $i = 1, 2, \dots, N$ and $\sum p_i \leq 1$.

- (i) When $\beta = 1$ and $\alpha \rightarrow 1$, (2.1) reduces to a measure of useful information for the incomplete distribution due to Belis and Giasu [3].
- (ii) When $u_i = 1$ for each i i.e., when the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$ and $\alpha \rightarrow 1$, the measure (2.1) reduces to Shannon's entropy [10].
- (iii) When $u_i = 1$ for each i , the measure (2.1) becomes entropy for the β -power distribution derived from P studied by Roy [9]. We call ${}_{\alpha\beta}H(U; P)$ in (2.1) the generalized useful measure of information for the incomplete power distribution P^β .

Further consider,

$$(2.2) \quad {}_{\alpha\beta}L_u = \frac{\alpha}{\alpha - 1} \left[1 - \sum P_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-n_i(\frac{\alpha-1}{\alpha})} \right],$$

where $\alpha > 0$ ($\neq 1$), $\sum p_i \leq 1$.



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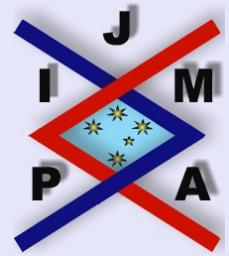


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- (i) For $\beta = 1$, $u_i = 1$ for each i and $\alpha \rightarrow 1$, $\alpha_\beta L_u$ in (2.2) reduces to the useful mean length L_u of the code given in (1.4).
- (ii) For $\beta = 1$, $u_i = 1$ for each i and $\alpha \rightarrow 1$, $\alpha_\beta L_u$ becomes the optimal code length defined by Shannon [10].

We establish a result, that in a sense, provides a characterization of $\alpha_\beta H(U; P)$ under the condition of unique decipherability.

Theorem 2.1. For all integers $D > 1$

$$(2.3) \quad \alpha_\beta L_u \geq \alpha_\beta H(U; P)$$

under the condition (1.3). Equality holds if and only if

$$(2.4) \quad n_i = -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right).$$

Proof. We use Hölder's [11] inequality

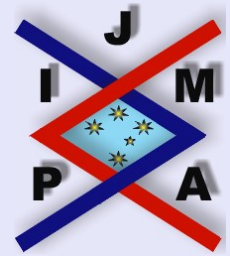
$$(2.5) \quad \sum x_i y_i \geq \left(\sum x_i^p \right)^{\frac{1}{p}} \left(\sum y_i^q \right)^{\frac{1}{q}}$$

for all $x_i \geq 0$, $y_i \geq 0$, $i = 1, 2, \dots, N$ when $P < 1$ ($\neq 1$) and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number c such that

$$(2.6) \quad x_i^p = c y_i^q.$$

Setting

$$x_i = p_i^{\frac{\alpha\beta}{\alpha-1}} \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha-1}} D^{-n_i},$$



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$$y_i = p_i^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{1-\alpha}},$$

$p = 1 - 1/\alpha$ and $q = 1 - \alpha$ in (2.5) and using (1.3) we obtain the result (2.3) after simplification for $\frac{\alpha}{\alpha-1} > 0$ as $\alpha > 1$. \square

Theorem 2.2. For every code with lengths $\{n_i\}$, $i = 1, 2, \dots, N$, ${}_{\alpha\beta}L_u$ can be made to satisfy,

$$(2.7) \quad {}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H(U; P)D^{\left(\frac{1-\alpha}{\alpha}\right)} + \frac{\alpha}{1-\alpha} \left[1 - D^{\left(\frac{1-\alpha}{\alpha}\right)} \right].$$

Proof. Let n_i be the positive integer satisfying, the inequality

$$(2.8) \quad -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) \leq n_i < -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) + 1.$$

Consider the intervals

$$(2.9) \quad \delta_i = \left[-\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right), -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) + 1 \right]$$

of length 1. In every δ_i , there lies exactly one positive number n_i such that

$$(2.10) \quad 0 < -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) \leq n_i < -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) + 1.$$

It can be shown that the sequence $\{n_i\}$, $i = 1, 2, \dots, N$ thus defined, satisfies (1.3). From (2.10) we have

$$\begin{aligned}
 (2.11) \quad n_i &< -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) + 1 \\
 &\Rightarrow D^{-n_i} < \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right) D \\
 &\Rightarrow D^{-n_i \left(\frac{\alpha-1}{\alpha} \right)} < \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right)^{\frac{1-\alpha}{\alpha}} D^{\frac{\alpha-1}{\alpha}}
 \end{aligned}$$

Multiplying both sides of (2.11) by $p_i^\beta \left(\frac{u_i}{\sum u_i p_i^{\alpha\beta}} \right)^{\frac{1}{\alpha}}$, summing over $i = 1, 2, \dots, N$ and simplifying, gives (2.7). □

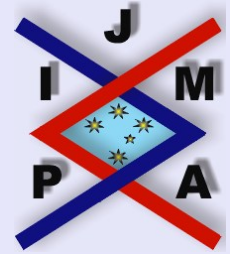
Theorem 2.3. For every code with lengths $\{n_i\}$, $i = 1, 2, \dots, N$, of Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisfy

$$(2.12) \quad {}_{\alpha\beta}H(U; P) \leq {}_{\alpha\beta}L_u < {}_{\alpha\beta}H(U; P) + \frac{\alpha}{\alpha-1}(1-D)$$

Proof. Suppose

$$(2.13) \quad \bar{n}_i = -\log \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}} \right)$$

Clearly \bar{n}_i and $\bar{n}_i + 1$ satisfy ‘equality’ in Hölder’s inequality (2.5). Moreover, \bar{n}_i satisfies Kraft’s inequality (1.3).



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Suppose n_i is the unique integer between \bar{n}_i and $\bar{n}_i + 1$, then obviously, n_i satisfies (1.3).

Since $\alpha > 0$ ($\neq 1$), we have

$$\begin{aligned}
 (2.14) \quad \sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{\bar{n}_i(\alpha-1)/\alpha} & \\
 & \leq \sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{n_i(\alpha-1)/\alpha} \\
 & < D \left(\sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{\bar{n}_i(\alpha-1)/\alpha} \right)
 \end{aligned}$$

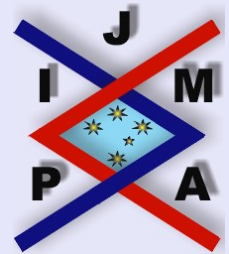
Since,

$$\sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{\bar{n}_i(\alpha-1)/\alpha} = \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}}$$

Hence, (2.14) becomes

$$\left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} \leq \sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-\bar{n}_i(\alpha-1)/\alpha} < D \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^\beta} \right)^{\frac{1}{\alpha}}$$

which gives the result (2.12). □



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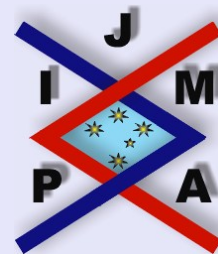
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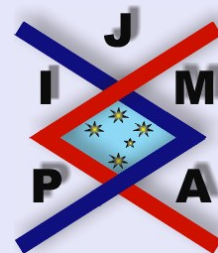
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