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SOME INEQUALITIES FOR THE q-DIGAMMA FUNCTION

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ABSTRACT. For the q-digamma function and it's derivatives are established the functional inequalities of the types:

$$f^2(x \cdot y) \lesssim f(x) \cdot f(y),$$
$$f(x+y) \lesssim f(x) + f(y).$$

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1. INTRODUCTION

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{1.1}$$

where $\gamma = 0.57721\ldots$ denotes Euler's constant. Another interesting series representation for ψ, which is "more rapidly convergent" than the one given in (1.1), was discovered by Ramanujan [3, page 374].

177-08

Jackson (see [5, 6, 7, 8]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1, \tag{1.2}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, \quad q>1, \tag{1.3}$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$.

The q-analogue of the psi function is defined for $0<q<1$ as the logarithmic derivative of the q-gamma function, that is,

$$\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x).$$

Many properties of the q-gamma function were derived by Askey [2]. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. From (1.2), for $0<q<1$ and $x>0$ we get

$$\begin{aligned} \psi_q(x) &= -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}} \\ &= -\log(1-q) + \log q \sum_{n\geq 1} \frac{q^{nx}}{1-q^{n}} \end{aligned} \tag{1.4}$$

and from (1.3) for $q>1$ and $x>0$ we obtain

$$\begin{aligned} \psi_q(x) &= -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 0} \frac{q^{-n-x}}{1-q^{-n-x}} \right) \\ &= -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 1} \frac{q^{-nx}}{1-q^{-n}} \right). \end{aligned} \tag{1.5}$$

A Stieltjes integral representation for $\psi_q(x)$ with $0<q<1$ is given in [4]. It is well-known that ψ' is strictly completely monotonic on $(0,\infty)$, that is,

$$(-1)^n(\psi'(x))^{(n)} > 0 \quad \text{for } x>0 \text{ and } n\geq 0,$$

see [1, Page 260]. From (1.4) and (1.5) we conclude that ψ_q' has the same property for any $q>0$

$$(-1)^n(\psi_q'(x))^{(n)} > 0 \quad \text{for } x>0 \text{ and } n\geq 0.$$

If $q\in(0,1)$, using the second representation of $\psi_q(x)$ given in (1.4), it can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k \cdot q^{nx}}{1-q^n} \tag{1.6}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x>1$, for all $k\geq 1$. If $q>1$, from the second representation of $\psi_q(x)$ given in (1.5), we obtain

$$\psi_q'(x) = \log q \left(1 + \sum_{n\geq 1} \frac{nq^{-nx}}{1-q^{-nx}} \right) \tag{1.7}$$

and for $k\geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1}\log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{-nx}}{1-q^{-nx}} \tag{1.8}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x>0$, for all $q>1$.

In this paper we derive several inequalities for $\psi^{(k)}(x)$, where $k \geq 0$.

2. INEQUALITIES OF THE TYPE $f^2(x \cdot y) \lesseqgtr f(x) \cdot f(y)$

We start with the following lemma.

Lemma 2.1. *For $0 < q < \frac{1}{2}$ and $0 < x < 1$ we have that $\psi_q(x) < 0$.*

Proof. At first let us prove that $\psi_q(x) < 0$ for all $x > 0$. From (1.4) we get that

$$\psi_q(x) = \frac{q^x}{1-q}\log q - \log(1-q) + \log q \sum_{n\geq 2} \frac{q^{nx}}{1-q^n}.$$

In order to see that $\psi_q(x) < 0$, we need to show that the function

$$g(x) = \frac{q^x}{1-q}\log q - \log(1-q)$$

is a negative for all $0 < x < 1$ and $0 < q < \frac{1}{2}$. Indeed $g'(x) = \frac{q^x}{1-q}\log^2 q > 0$, which implies that $g(x)$ is an increasing function on $0 < x < 1$, hence

$$\begin{aligned} g(x) < g(1) &= \frac{q}{1-q}\log q - \log(1-q) \\ &= \frac{1}{1-q}\log\frac{q^q}{(1-q)^{1-q}} < 0, \end{aligned}$$

for all $0 < q < \frac{1}{2}$. □

Theorem 2.2. *Let $0 < q < \frac{1}{2}$ and $0 < x, y < 1$. Let $k \geq 0$ be an integer. Then*

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) < (\psi_q^{(k)}(xy))^2.$$

Proof. We will consider two different cases: (1) $k = 0$ and (2) $k \geq 1$.
(1) Let $f(x) = \psi_q^2(x)$ defined on $0 < x < 1$. By Lemma 2.1 we have that

$$f'(x) = 2\psi_q(x)\psi_q'(x) < 0$$

for all $0 < x < 1$, which gives that $f(x)$ is a decreasing function on $0 < x < 1$. Hence, for all $0 < x, y < 1$ we have

$$\psi_q^2(xy) > \psi_q^2(x) \quad \text{and} \quad \psi_q^2(xy) > \psi_q^2(y),$$

which gives that

$$\psi_q^4(xy) > \psi_q^2(x)\psi_q^2(y).$$

Since $\psi_q(x)\psi_q(y) > 0$ for all $0 < x, y < 1$, see Lemma 2.1, we obtain that

$$\psi_q^2(xy) > \psi_q(x)\psi_q(y),$$

as claimed.

(2) From (1.6) we have that

$$\begin{aligned}
&\psi_q^{(k)}(x)\psi_q^{(k)}(y)-(\psi_q^{(k)}(xy))^2\\
&=\left(\log^{k+1}q\sum_{n\geq 1}\frac{n^kq^{nx}}{1-q^n}\right)\left(\log^{k+1}q\sum_{n\geq 1}\frac{n^kq^{ny}}{1-q^n}\right)-\left(\log^{k+1}q\sum_{n\geq 1}\frac{n^kq^{nxy}}{1-q^n}\right)^2\\
&=(\log^{k+1}q)^2\sum_{n,m\geq 1}\frac{n^kq^{nx}}{1-q^n}\cdot\frac{m^kq^{my}}{1-q^m}-(\log^{k+1}q)^2\sum_{n,m\geq 1}\frac{(nm)^kq^{(n+m)xy}}{(1-q^n)(1-q^m)}\\
&=(\log^{k+1}q)^2\sum_{n,m\geq 1}\frac{(nm)^k(q^{nx+my}-q^{(n+m)xy})}{(1-q^n)(1-q^m)}.
\end{aligned}$$

For $0<x,y<1$, $q^{nx+my}-q^{(n+m)xy}<0$ and for $x,y>1$, $q^{nx+my}-q^{(n+m)xy}>0$ and the results follow. □

Note that the above theorem for $k\geq 1$ remains true also for $q\in\left[\frac{1}{2},1\right]$. Also, if $x,y>1$, $k\geq 1$ and $0<q<1$ then

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y)>\left(\psi_q^{(k)}(xy)\right)^2.$$

Now we extend Lemma 2.1 to the case $q>1$. In order to do that we denote the zero of the function $f(q)=\frac{q-3}{2(q-1)}\log(q)-\log(q-1)$, $q>1$, by q^*. The numerical solution shows that $q^*\approx 1.56683201\ldots$ as shown on Figure 2.1.

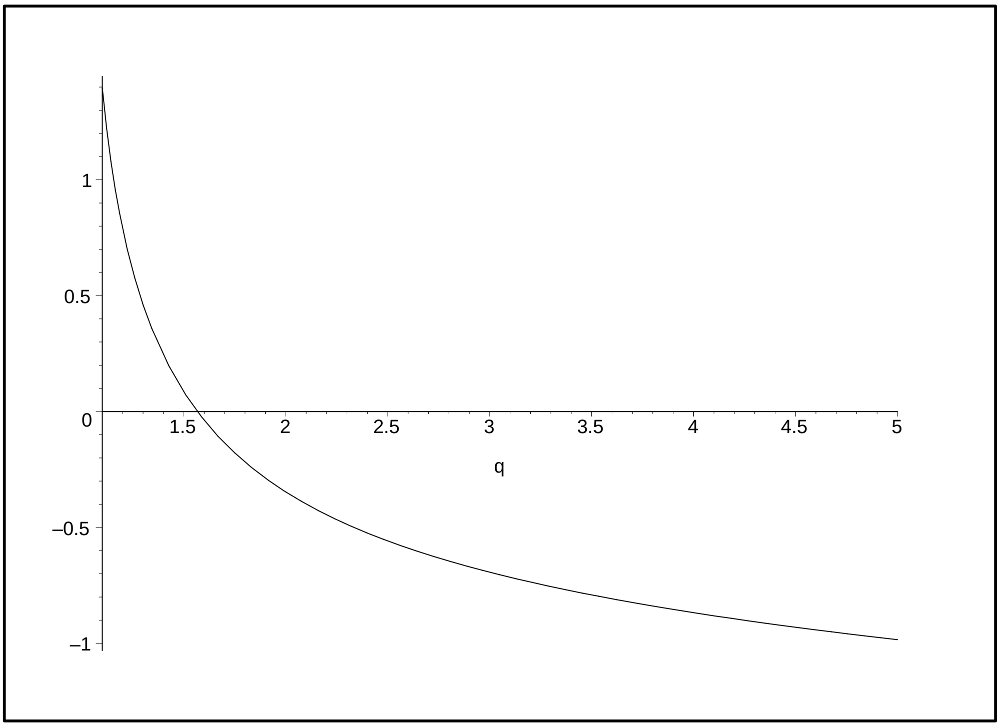

Figure 2.1: Graph of the function $\frac{q-3}{2(q-1)}\log q-\log(q-1)$.

Lemma 2.3. *For* $q>q^*$ *and* $0<x<1$ *we have that* $\psi_q(x)<0$.

Proof. From (1.5) we get that

$$\psi_q(x)=-\frac{q^{-x}}{1-q^{-1}}\log q-\log(q-1)+\log q\left(x-\frac{1}{2}\right)-\log q\sum_{n\geq 2}\frac{q^{-nx}}{1-q^{-n}}.$$

In order to show our claim, we need to prove that

$$g(x) = -\frac{q^{-x}}{1-q^{-1}} \log q - \log(q-1) + \log q \left(x - \frac{1}{2} \right) < 0$$

on $0 < x < 1$. Since $g'(x) = \frac{q^{-x}}{1-q^{-1}} \log^2 q + \log q > 0$, it implies that $g(x)$ is an increasing function on $0 < x < 1$. Hence

$$g(x) < g(1) = \frac{q-3}{2(q-1)} \log q - \log(q-1) < 0,$$

for all $q > q^*$, see Figure 2.1. □

Theorem 2.4. *Let* $q > 2$ *and* $0 < x, y < 1$. *Let* $k \geq 0$ *be an integer. Then*

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) < \left(\psi_q^{(k)}(xy)\right)^2.$$

Proof. As in the previous theorem we will consider two different cases: (1) $k = 0$ and (2) $k \geq 1$.
(1) As shown in the introduction the function $\psi_q'(x)$ is an increasing function on $0 < x < 1$. Therefore, for all $0 < x, y < 1$ we have that

$$\psi_q(xy) < \psi_q(x) \quad \text{and} \quad \psi_q(xy) < \psi_q(y).$$

Hence, Lemma 2.3 gives that $\psi_q^2(xy) > \psi_q(x)\psi_q(y)$, as claimed.

(2) Analogous to the second case of Theorem 2.2. □

Note that Theorem 2.4 for $k \geq 1$ remains true also for $q > 1$. Also, if $x, y > 1$, $k \geq 1$ and $q > 1$ then

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) > \left(\psi_q^{(k)}(xy)\right)^2.$$

3. INEQUALITIES OF THE TYPE $f(x+y) \lesseqgtr f(x) + f(y)$

The main goal of this section is to show that $\psi_q(x+y) \geq \psi_q(x) + \psi_q(y)$, for all $0 < x, y < 1$ and $0 < q < 1$. In order to do that we define

$$\rho(q) = \log(1-q) + \log q \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j}.$$

Lemma 3.1. *For all* $0 < q < 1$, $\rho(q) > 0$.

Proof. Let $0 < q < 1$ and let $g_m(q) = c + \sum_{j=1}^{m-1} \frac{q^j(q^j-2)}{1-q^j}$ with constant $c > 0$ for $m \geq 2$. Then $g_m(0) = c$, $\lim_{q\to 1^-} g_m(q) < 0$ and $g_m(q)$ is a decreasing function since

$$g_m'(q) = -\sum_{j=1}^{m-1} \frac{jq^{j-1}(1+(1-q^j)^2)}{(1-q^j)^2} < 0.$$

On the other hand

$$g_{m+1}(q) - g_m(q) = \frac{q^m(q^m-2)}{1-q^m} < 0$$

for all $0 < q < 1$. Hence, for all $m \geq 2$ we have that

$$g_{m+1}(q) < g_m(q), \quad 0 < q < 1.$$

Thus, if b_m is the positive zero of the function $g_m(q)$ (because $g_n(q)$ is decreasing) on $0 < q < 1$ (by Maple or any mathematical programming we can see that $b_1 = 0.38196601\ldots$, $b_2 = 0.3184588966$ and $b_3 = 0.3055970874$), then $g_m(q) > 0$ for all $0 < q < b_m$ and $g_m(q) < 0$

for all $b_m < q < 1$. Furthermore, the sequence $\{b_m\}_{m\geq 0}$ is a strictly decreasing sequence of positive real numbers, that is $0 < b_{m+1} < b_m$, and bounded by zero, which implies that

$$\lim_{m\to\infty} g_m(q) = c + \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j} < 0,$$

for all $0 < q < 1$. Hence, if we choose $c = \frac{2\log(1-q)}{\log q}$ (c is positive since $0 < q < 1$), then we have that

$$\sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j} < -\frac{2\log(1-q)}{\log q},$$

which implies that

$$\begin{aligned}\rho(q) &= \log(1-q) + \log q \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j}\\ &> -2\log(1-q) + \log(1-q)\\ &= -\log(1-q) > 0,\end{aligned}$$

as requested. □

Theorem 3.2. *For all* $0 < q < 1$ *and* $0 < x, y < 1$*,*

$$\psi_q(x+y) > \psi_q(x) + \psi_q(y).$$

Proof. From the definitions we have that

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y) = \log(1-q) + \log q \sum_{n\geq 1} \frac{q^{n(x+y)} - q^{nx} - q^{ny}}{1-q^n}.$$

Since $0 < x, y, q < 1$, we have that

$$\begin{aligned}q^{n(x+y)} - q^{nx} - q^{ny} &= (1-q^{nx})(1-q^{ny}) - 1\\ &< (1-q^n)^2 - 1\\ &= q^n(q^n-2).\end{aligned}$$

Hence, by Lemma 3.1

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y) > \rho(q) > 0,$$

which completes the proof. □

The above theorem is not true for $x, y > 1$, for example

$$\begin{aligned}&\psi_{1/10}(4) = 0.1051046497\ldots, \quad \psi_{1/10}(5) = 0.1053349312\ldots,\\ &\psi_{1/10}(9) = 0.1053605131\ldots.\end{aligned}$$

Theorem 3.3. *For all* $q > 1$ *and* $0 < x, y < 1$*,*

$$\psi_q(x+y) > \psi_q(x) + \psi_q(y).$$

Proof. From the definitions we have that

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y) = \log(q-1) + \frac{1}{2}\log q + \log Q \sum_{n\geq 1} \frac{Q^{n(x+y)} - Q^{nx} - Q^{ny}}{1-Q^n},$$

where $Q = 1/q$. Thus

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y)$$
$$= \log(q-1) + \frac{1}{2}\log q + \psi_Q(x+y) - \psi_Q(x) - \psi_Q(y) - \log(1-Q).$$

Using Theorem 3.2 we get that

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y) > \log(q-1) + \frac{1}{2}\log q - \log(q-1) + \log q > 0,$$

which completes the proof. □

Note that the above theorem holds for $q > 2$ and $x, y > 1$, since

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y)$$
$$= \log(q-1) + \frac{1}{2}\log q + \log q \sum_{n\geq 1} \frac{q^{-nx}(1-q^{-ny}) + q^{-ny}}{1-q^{-n}} > 0.$$

The above theorem is not true for $x, y > 1$ when $1 < q < 2$, for example

$$\psi_{3/2}(4) = 1.83813910\ldots, \quad \psi_{3/2}(5) = 2.34341101\ldots,$$
$$\psi_{3/2}(9) = 4.10745515\ldots.$$

Theorem 3.4. *Let* $q \in (0,1)$. *Let* $k \geq 1$ *be an integer.*

(1) *If* k *is even then*

$$\psi_q^{(k)}(x+y) \geq \psi_q^{(k)}(x) + \psi_q^{(k)}(y).$$

(2) *If* k *is odd then*

$$\psi_q^{(k)}(x+y) \leq \psi_q^{(k)}(x) + \psi_q^{(k)}(y).$$

Proof. From (1.6) we have

$$\psi_q^k(x+y) - \psi_q^k(x) - \psi_q^k(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k}{1-q^n}(q^{n(x+y)} - q^{nx} - q^{ny}).$$

Since the function $f(z) = q^{nz}$ is convex from

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)),$$

we obtain that

$$2 \cdot q^{n\frac{x+y}{2}} \leq q^{nx} + q^{ny}. \tag{3.1}$$

On the other hand it is clear that

$$2 \cdot q^{n\frac{x+y}{2}} > q^{n(x+y)}. \tag{3.2}$$

From (3.1) and (3.2) we have that

$$q^{n(x+y)} - q^{nx} - q^{ny} < 0.$$

(1) Since for $q \in (0,1)$ and k even we have $\log^{k+1} q < 0$, hence

$$\psi_q^{(k)}(x+y) - \psi_q^{(k)}(x) - \psi_q^{(k)}(x) \geq 0.$$

(2) The other case can be proved in a similar manner. □

Using a similar approach one may prove analogue results for $q > 1$.

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