

# Journal of Inequalities in Pure and Applied Mathematics

## LITTLEWOOD-PALEY $g$ -FUNCTION IN THE DUNKL ANALYSIS ON $\mathbb{R}^d$

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©2000 Victoria University  
ISSN (electronic): 1443-5756  
183-04



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volume 6, issue 3, article 84,  
2005.

*Received 11 October, 2004;  
accepted 22 July, 2005.*

*Communicated by: S.S. Dragomir*

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Abstract

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## Abstract

We prove  $L^p$ -inequality for the Littlewood-Paley  $g$ -function in the Dunkl case on  $\mathbb{R}^d$ .

*2000 Mathematics Subject Classification:* 42B15, 42B25.

*Key words:* Dunkl operators, Generalized Poisson integral,  $g$ -function.

The author is very grateful to the referee for many comments on this paper.

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# 1. Introduction

In the Euclidean case, the Littlewood-Paley  $g$ -function is given by

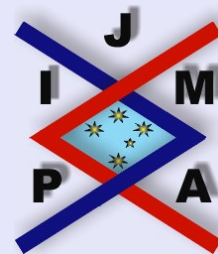
$$g(f)(x) := \left[ \int_0^\infty \left( \left| \frac{\partial}{\partial t} u(x, t) \right|^2 + |\nabla_x u(x, t)|^2 \right) t dt \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where  $u$  is the Poisson integral of  $f$  and  $\nabla$  is the usual gradient. The  $L^p$ -norm of this operator is comparable with the  $L^p$ -norm of  $f$  for  $p \in ]1, \infty[$  (see [19]). Next, this operator plays an important role in questions related to multipliers, Sobolev spaces and Hardy spaces (see [19]).

Over the past twenty years considerable effort has been made to extend the Littlewood-Paley  $g$ -function on generalized hypergroups [20, 1, 2], and complete Riemannian manifolds [4].

In this paper we consider the differential-difference operators  $T_j; j = 1, \dots, d$ , on  $\mathbb{R}^d$  introduced by Dunkl in [5] and aptly called Dunkl operators in the literature. These operators extend the usual partial derivatives by additional reflection terms and give generalizations of many multi-variable analytic structures like the exponential function, the Fourier transform, the convolution product and the Poisson integral (see [12, 23, 16] and [13]).

During the last years, these operators have gained considerable interest in various fields of mathematics and in certain parts of quantum mechanics; one expects that the results in this paper will be useful when discussing the boundedness property of the Littlewood-Paley  $g$ -function in the Dunkl analysis on  $\mathbb{R}^d$ . Moreover they are naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many body systems [3, 9].



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The main purpose of this paper is to give the  $L^p$ -inequality for the Littlewood-Paley  $g$ -function in the Dunkl case on  $\mathbb{R}^d$  by using continuity properties of the Dunkl transform  $\mathcal{F}_k$ , the Dunkl translation operators of radial functions and the generalized convolution product  $*_k$ . We will adapt to this case techniques Stein used in [18, 19].

The paper is organized as follows. In Section 2 we recall some basic harmonic analysis results related to the Dunkl operators on  $\mathbb{R}^d$ . In particular, we list some basic properties of the Dunkl transform  $\mathcal{F}_k$  and the generalized convolution product  $*_k$  (see [8, 23, 15]).

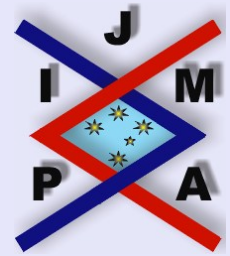
In Section 3 we study the Littlewood-Paley  $g$ -function:

$$g(f)(x) := \left[ \int_0^\infty \left( \left| \frac{\partial}{\partial t} u_k(x, t) \right|^2 + |\nabla_x u_k(x, t)|^2 \right) t dt \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where  $u_k(\cdot, t)$  is the generalized Poisson integral of  $f$ .

We prove that  $g$  is  $L^p$ -boundedness for  $p \in ]1, 2]$ .

Throughout the paper  $c$  denotes a positive constant whose value may vary from line to line.




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## 2. The Dunkl Analysis on $\mathbb{R}^d$

We consider  $\mathbb{R}^d$  with the Euclidean inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|x\| = \sqrt{\langle x, x \rangle}$ .

For  $\alpha \in \mathbb{R}^d \setminus \{0\}$ , let  $\sigma_\alpha$  be the reflection in the hyperplane  $H_\alpha \subset \mathbb{R}^d$  orthogonal to  $\alpha$ :

$$\sigma_\alpha x := x - \left( \frac{2\langle \alpha, x \rangle}{\|\alpha\|^2} \right) \alpha.$$

A finite set  $R \subset \mathbb{R}^d \setminus \{0\}$  is called a root system, if  $R \cap \mathbb{R} \alpha = \{-\alpha, \alpha\}$  and  $\sigma_\alpha R = R$  for all  $\alpha \in R$ . We assume that it is normalized by  $\|\alpha\|^2 = 2$  for all  $\alpha \in R$ .

For a root system  $R$ , the reflections  $\sigma_\alpha$ ,  $\alpha \in R$  generate a finite group  $G \subset O(d)$ , the reflection group associated with  $R$ . All reflections in  $G$ , correspond to suitable pairs of roots. For a given  $\beta \in H := \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$ , we fix the positive subsystem:

$$R_+ := \{\alpha \in R / \langle \alpha, \beta \rangle > 0\}.$$

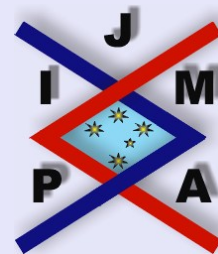
Then for each  $\alpha \in R$  either  $\alpha \in R_+$  or  $-\alpha \in R_+$ .

Let  $k : R \rightarrow \mathbb{C}$  be a multiplicity function on  $R$  (i.e. a function which is constant on the orbits under the action of  $G$ ). For brevity, we introduce the index:

$$\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha).$$

Moreover, let  $w_k$  denote the weight function:

$$w_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,$$




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which is  $G$ -invariant and homogeneous of degree  $2\gamma$ .

We introduce the Mehta-type constant  $c_k$ , by

$$(2.1) \quad c_k := \left( \int_{\mathbb{R}^d} e^{-\|x\|^2} d\mu_k(x) \right)^{-1}, \quad \text{where} \quad d\mu_k(x) := w_k(x)dx.$$

The Dunkl operators  $T_j$ ;  $j = 1, \dots, d$ , on  $\mathbb{R}^d$  associated with the finite reflection group  $G$  and multiplicity function  $k$  are given for a function  $f$  of class  $C^1$  on  $\mathbb{R}^d$ , by

$$T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

The generalized Laplacian  $\Delta_k$  associated with  $G$  and  $k$ , is defined by  $\Delta_k := \sum_{j=1}^d T_j^2$ . It is given explicitly by

$$(2.2) \quad \Delta_k f(x) := L_k f(x) - 2 \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2},$$

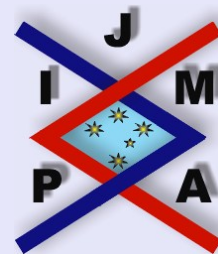
with the singular elliptic operator:

$$(2.3) \quad L_k f(x) := \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle},$$

where  $\Delta$  denotes the usual Laplacian.

The operator  $L_k$  can also be written in divergence form:

$$(2.4) \quad L_k f(x) = \frac{1}{w_k(x)} \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( w_k(x) \frac{\partial}{\partial x_i} \right).$$



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This is a canonical multi-variable generalization of the Sturm-Liouville operator for the classical spherical Bessel function [1, 2, 20].

For  $y \in \mathbb{R}^d$ , the initial value problem  $T_j u(x, \cdot)(y) = x_j u(x, y); j = 1, \dots, d$ , with  $u(0, y) = 1$  admits a unique analytic solution on  $\mathbb{R}^d$ , which will be denoted by  $E_k(x, y)$  and called a Dunkl kernel [6, 14, 16, 23].

This kernel has the Bochner-type representation (see [12]):

$$(2.5) \quad E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\Gamma_x(y); \quad x \in \mathbb{R}^d, z \in \mathbb{C}^d,$$

where  $\langle y, z \rangle := \sum_{i=1}^d y_i z_i$  and  $\Gamma_x$  is a probability measure on  $\mathbb{R}^d$  with support in the closed ball  $B_d(o, \|x\|)$  of center  $o$  and radius  $\|x\|$ .

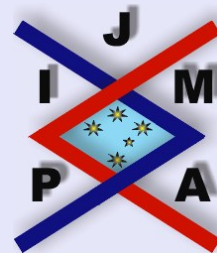
**Example 2.1 (see [23, p. 21]).** If  $G = \mathbb{Z}_2$ , the Dunkl kernel is given by

$$E_\gamma(x, z) = \frac{\Gamma(\gamma + \frac{1}{2})}{\sqrt{\pi} \Gamma(\gamma)} \cdot \frac{\text{sgn}(x)}{|x|^{2\gamma}} \int_{-|x|}^{|x|} e^{yz} (x^2 - y^2)^{\gamma-1} (x + y) dy.$$

**Notation.** We denote by  $\mathcal{D}(\mathbb{R}^d)$  the space of  $C^\infty$ -functions on  $\mathbb{R}^d$  with compact support.

The Dunkl kernel gives an integral transform, called the Dunkl transform on  $\mathbb{R}^d$ , which was studied by de Jeu in [8]. The Dunkl transform of a function  $f$  in  $\mathcal{D}(\mathbb{R}^d)$  is given by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$




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Note that  $\mathcal{F}_0$  agrees with the Fourier transform  $\mathcal{F}$  on  $\mathbb{R}^d$ :

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} e^{-i\langle x,y \rangle} f(y) dy, \quad x \in \mathbb{R}^d.$$

The Dunkl transform of a function  $f \in \mathcal{D}(\mathbb{R}^d)$  which is radial is again radial, and could be computed via the associated Fourier-Bessel transform  $\mathcal{F}_{\gamma+d/2-1}^B$  [11, p. 586] that is:

$$\mathcal{F}_k(f)(x) = 2^{\gamma+d/2} c_k^{-1} \mathcal{F}_{\gamma+d/2-1}^B(F)(\|x\|),$$

where  $f(x) = F(\|x\|)$ , and

$$\mathcal{F}_{\gamma+d/2-1}^B(F)(\|x\|) := \int_0^\infty F(r) \frac{j_{\gamma+d/2-1}(\|x\|r)}{2^{\gamma+d/2-1} \Gamma(\gamma + \frac{d}{2})} r^{2\gamma+d-1} dr.$$

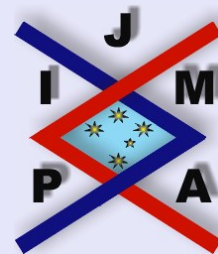
Here  $j_\gamma$  is the spherical Bessel function [24].

**Notations.** We denote by  $L_k^p(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , the space of measurable functions  $f$  on  $\mathbb{R}^d$ , such that

$$\|f\|_{L_k^p} := \left[ \int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x) \right]^{\frac{1}{p}} < \infty, \quad p \in [1, \infty[,$$

$$\|f\|_{L_k^\infty} := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty,$$

where  $\mu_k$  is the measure given by (2.1).



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## Theorem 2.1 (see [7]).

i) **Plancherel theorem:** the normalized Dunkl transform  $2^{-\gamma-d/2} c_k \mathcal{F}_k$  is an isometric automorphism on  $L_k^2(\mathbb{R}^d)$ . In particular,

$$\|f\|_{L_k^2} = 2^{-\gamma-d/2} c_k \|\mathcal{F}_k(f)\|_{L_k^2}.$$

ii) **Inversion formula:** let  $f$  be a function in  $L_k^1(\mathbb{R}^d)$ , such that  $\mathcal{F}_k(f) \in L_k^1(\mathbb{R}^d)$ . Then

$$\mathcal{F}_k^{-1}(f)(x) = 2^{-2\gamma-d} c_k^2 \mathcal{F}_k(f)(-x), \quad a.e. x \in \mathbb{R}^d.$$

In [6], Dunkl defines the intertwining operator  $V_k$  on  $\mathcal{P} := \mathbb{C}[\mathbb{R}^d]$  (the  $\mathbb{C}$ -algebra of polynomial functions on  $\mathbb{R}^d$ ), by

$$V_k(p)(x) := \int_{\mathbb{R}^d} p(y) d\Gamma_x(y), \quad x \in \mathbb{R}^d,$$

where  $\Gamma_x$  is the representing measure on  $\mathbb{R}^d$  given by (2.5).

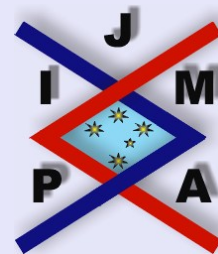
Next, Rösler proved the positivity properties of this operator (see [12]).

**Notation.** We denote by  $\mathcal{E}(\mathbb{R}^d)$  and by  $\mathcal{E}'(\mathbb{R}^d)$  the spaces of  $C^\infty$ -functions on  $\mathbb{R}^d$  and of distributions on  $\mathbb{R}^d$  with compact support respectively.

In [22, Theorem 6.3], Trimèche has proved the following results:

## Proposition 2.2.

i) The operator  $V_k$  can be extended to a topological automorphism on  $\mathcal{E}(\mathbb{R}^d)$ .



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ii) For all  $x \in \mathbb{R}^d$ , there exists a unique distribution  $\eta_{k,x}$  in  $\mathcal{E}'(\mathbb{R}^d)$  with  $\text{supp}(\eta_{k,x}) \subset \{y \in \mathbb{R}^d / \|y\| \leq \|x\|\}$ , such that

$$(V_k)^{-1}(f)(x) = \langle \eta_{k,x}, f \rangle, \quad f \in \mathcal{E}(\mathbb{R}^d).$$

Next in [23], the author defines:

- The Dunkl translation operators  $\tau_x$ ,  $x \in \mathbb{R}^d$ , on  $\mathcal{E}(\mathbb{R}^d)$ , by

$$\tau_x f(y) := (V_k)_x \otimes (V_k)_y [(V_k)^{-1}(f)(x + y)], \quad y \in \mathbb{R}^d.$$

These operators satisfy for  $x, y$  and  $z \in \mathbb{R}^d$  the following properties:

$$(2.6) \quad \tau_0 f = f, \quad \tau_x f(y) = \tau_y f(x),$$

$$E_k(x, z)E_k(y, z) = \tau_x(E_k(\cdot, z))(x),$$

and

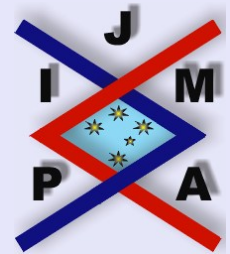
$$(2.7) \quad \mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y), \quad f \in \mathcal{D}(\mathbb{R}^d).$$

Thus by (2.7), the Dunkl translation operators can be extended on  $L_k^2(\mathbb{R}^d)$ , and for  $x \in \mathbb{R}^d$  we have

$$\|\tau_x f\|_{L_k^2} \leq \|f\|_{L_k^2}, \quad f \in L_k^2(\mathbb{R}^d).$$

- The generalized convolution product  $*_k$  of two functions  $f$  and  $g$  in  $L_k^2(\mathbb{R}^d)$ , by

$$f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y)g(y)d\mu_k(y), \quad x \in \mathbb{R}^d.$$



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Note that  $*_0$  agrees with the standard convolution  $*$  on  $\mathbb{R}^d$ :

$$f * g(x) := \int_{\mathbb{R}^d} f(x - y)g(y)dy, \quad x \in \mathbb{R}^d.$$

The generalized convolution  $*_k$  satisfies the following properties:

**Proposition 2.3.**

i) Let  $f, g \in \mathcal{D}(\mathbb{R}^d)$ . Then

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g).$$

ii) Let  $f, g \in L_k^2(\mathbb{R}^d)$ . Then  $f *_k g$  belongs to  $L_k^2(\mathbb{R}^d)$  if and only if  $\mathcal{F}_k(f)\mathcal{F}_k(g)$  belongs to  $L_k^2(\mathbb{R}^d)$  and we have

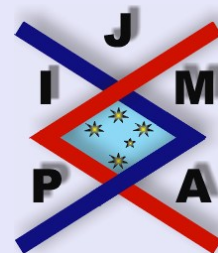
$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g), \quad \text{in the } L_k^2 \text{ - case.}$$

*Proof.* The assertion i) is shown in [23, Theorem 7.2]. We can prove ii) in the same manner demonstrated in [21, p. 101–103]. □

**Theorem 2.4.** Let  $p, q, r \in [1, \infty]$  satisfy the Young’s condition:  $1/p + 1/q = 1 + 1/r$ . Assume that  $f \in L_k^p(\mathbb{R}^d)$  and  $g \in L_k^q(\mathbb{R}^d)$ . If  $\|\tau_x f\|_{L_k^q} \leq c \|f\|_{L_k^q}$  for all  $x \in \mathbb{R}^d$ , then

$$\|f *_k g\|_{L_k^r} \leq c \|f\|_{L_k^p} \|g\|_{L_k^q}.$$

*Proof.* The assumption that  $\tau_x$  is a bounded operator on  $L_k^q(\mathbb{R}^d)$  ensures that the usual proof of Young’s inequality (see [25, p. 37]) works. □



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**Proposition 2.5.**

i) If  $f(x) = F(\|x\|)$  in  $\mathcal{E}(\mathbb{R}^d)$ , then we have

$$\tau_x f(y) = \int_{\mathcal{A}_{x,y}} F\left(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle y, \xi \rangle}\right) d\Gamma_x(\xi); \quad x, y \in \mathbb{R}^d,$$

where

$$\mathcal{A}_{x,y} = \left\{ \xi \in \mathbb{R}^d / \min_{g \in G} \|x + gy\| \leq \|\xi\| \leq \max_{g \in G} \|x + gy\| \right\},$$

and  $\Gamma_x$  the representing measure given by (2.5).

ii) For all  $x \in \mathbb{R}^d$  and for  $f \in L_k^p(\mathbb{R}^d)$ , radial,  $p \in [1, \infty]$ ,

$$\|\tau_x f\|_{L_k^p} \leq \|f\|_{L_k^p}.$$

iii) Let  $p, q, r \in [1, \infty]$  satisfy the Young's condition:  $1/p + 1/q = 1 + 1/r$ . Assume that  $f \in L_k^p(\mathbb{R}^d)$ , radial, and  $g \in L_k^q(\mathbb{R}^d)$ , then

$$\|f *_k g\|_{L_k^r} \leq \|f\|_{L_k^p} \|g\|_{L_k^q}.$$

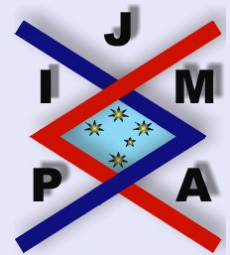
*Proof.* The assertion i) is shown by Rösler in [13, Theorem 5.1].

ii) Since  $f$  is a radial function, the explicit formula of  $\tau_x f$  shows that

$$|\tau_x f(y)| \leq \tau_x(|f|)(y).$$

Hence, it follows readily from (2.6) that

$$\|\tau_x f\|_{L_k^1} \leq \|f\|_{L_k^1}.$$



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By duality the same inequality holds for  $p = \infty$ .

Thus by interpolation we obtain the result for  $p \in ]1, \infty[$ .

iii) follows directly from Theorem 2.4. □

**Notation.** For all  $x, y, z \in \mathbb{R}$ , we put

$$W_\gamma(x, y, z) := [1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}] B_\gamma(|x|, |y|, |z|),$$

where

$$\sigma_{x,y,z} := \begin{cases} \frac{x^2 + y^2 - z^2}{2xy}, & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ 0, & \text{otherwise} \end{cases}$$

and  $B_\gamma$  is the Bessel kernel given by

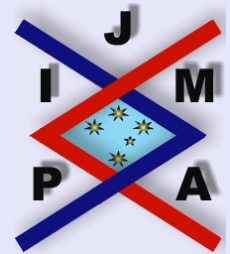
$$B_\gamma(|x|, |y|, |z|) := \begin{cases} d_\gamma \frac{[( (|x| + |y|)^2 - z^2 ) (z^2 - (|x| - |y|)^2)]^{\gamma-1}}{|xyz|^{2\gamma-1}}, & \text{if } |z| \in A_{x,y} \\ 0, & \text{otherwise,} \end{cases}$$

$$d_\gamma = \frac{2^{-2\gamma+1} \Gamma(\gamma + \frac{1}{2})}{\sqrt{\pi} \Gamma(\gamma)}, \quad A_{x,y} = [ | |x| - |y| |, |x| + |y| ].$$

**Remark 1 (see [10]).** The signed kernel  $W_\gamma$  is even and satisfies:

$$W_\gamma(x, y, z) = W_\gamma(y, x, z) = W_\gamma(-x, z, y),$$

$$W_\gamma(x, y, z) = W_\gamma(-z, y, -x) = W_\gamma(-x, -y, -z),$$



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and

$$\int_{\mathbb{R}} |W_{\gamma}(x, y, z)| dz \leq 4.$$

We consider the signed measures  $\nu_{x,y}$  (see [10]) defined by

$$d\nu_{x,y}(z) := \begin{cases} W_{\gamma}(x, y, z)|z|^{2\gamma} dz, & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0. \end{cases}$$

The measures  $\nu_{x,y}$  have the following properties:

$$\text{supp}(\nu_{x,y}) = A_{x,y} \cup (-A_{x,y}), \quad \|\nu_{x,y}\| := \int_{\mathbb{R}} d|\nu_{x,y}| \leq 4.$$

**Proposition 2.6** (see [10, 15]). *If  $d = 1$  and  $G = \mathbb{Z}_2$ , then*

i) *For all  $x, y \in \mathbb{R}$  and for  $f$  a continuous function on  $\mathbb{R}$ , we have*

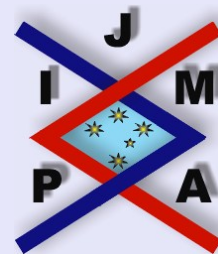
$$\tau_x f(y) = \int_{A_{x,y}} f(\xi) d\nu_{x,y}(\xi) + \int_{(-A_{x,y})} f(\xi) d\nu_{x,y}(\xi).$$

ii) *For all  $x \in \mathbb{R}$  and for  $f \in L^p_{\gamma}(\mathbb{R})$ ,  $p \in [1, \infty]$ ,*

$$\|\tau_x f\|_{L^p_{\gamma}} \leq 4 \|f\|_{L^p_{\gamma}}.$$

iii) *Assume that  $p, q, r \in [1, \infty]$  satisfy the Young's condition:  $1/p + 1/q = 1 + 1/r$ . Then the map  $(f, g) \rightarrow f *_{\gamma} g$  extends to a continuous map from  $L^p_{\gamma}(\mathbb{R}) \times L^q_{\gamma}(\mathbb{R})$  to  $L^r_{\gamma}(\mathbb{R})$  and we have*

$$\|f *_{\gamma} g\|_{L^r_{\gamma}} \leq 4 \|f\|_{L^p_{\gamma}} \|g\|_{L^q_{\gamma}}.$$



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### 3. The Littlewood-Paley $g$ -Function

By analogy with the case of Euclidean space [19, p. 61] we define, for  $t > 0$ , the functions  $W_t$  and  $P_t$  on  $\mathbb{R}^d$ , by

$$W_t(x) := 2^{-2\gamma-d} c_k^2 \int_{\mathbb{R}^d} e^{-t\|\xi\|^2} E_k(ix, \xi) d\mu_k(\xi), \quad x \in \mathbb{R}^d,$$

and

$$P_t(x) := 2^{-2\gamma-d} c_k^2 \int_{\mathbb{R}^d} e^{-t\|\xi\|} E_k(ix, \xi) d\mu_k(\xi), \quad x \in \mathbb{R}^d.$$

The function  $W_t$ , may be called the generalized heat kernel and the function  $P_t$ , the generalized Poisson kernel respectively.

From [23, p. 37] we have

$$W_t(x) = \frac{c_k}{(4t)^{\gamma+d/2}} e^{-\|x\|^2/4t}, \quad x \in \mathbb{R}^d.$$

Writing

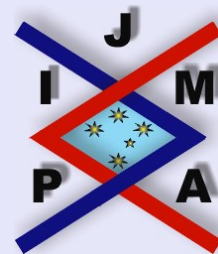
$$(3.1) \quad P_t(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} W_{t^2/4s}(x) ds, \quad x \in \mathbb{R}^d,$$

we obtain

$$(3.2) \quad P_t(x) = \frac{a_k t}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}}, \quad a_k := \frac{c_k \Gamma\left(\gamma + \frac{d+1}{2}\right)}{\sqrt{\pi}}.$$

However, for  $t > 0$  and for all  $f \in L_k^p(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , we put:

$$u_k(x, t) := P_t *_k f(x), \quad x \in \mathbb{R}^d.$$



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The function  $u_k$  is called the generalized Poisson integral of  $f$ , which was studied by Rösler in [11, 13].

Let us consider the Littlewood-Paley  $g$ -function (in the Dunkl case). This auxiliary operator is defined initially for  $f \in \mathcal{D}(\mathbb{R}^d)$ , by

$$g(f)(x) := \left[ \int_0^\infty \left( \left| \frac{\partial}{\partial t} u_k(x, t) \right|^2 + |\nabla_x u_k(x, t)|^2 \right) t dt \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where  $u_k$  is the generalized Poisson integral.

The main result of the paper is:

**Theorem 3.1.** For  $p \in ]1, 2]$ , there exists a constant  $A_p > 0$  such that, for  $f \in L_k^p(\mathbb{R}^d)$ ,

$$\|g(f)\|_{L_k^p} \leq A_p \|f\|_{L^p}.$$

For the proof of this theorem we need the following lemmas:

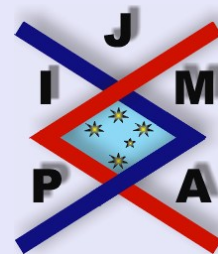
**Lemma 3.2.** Let  $f \in \mathcal{D}(\mathbb{R}^d)$  be a positive function.

i)  $u_k(x, t) \geq 0$  and  $\left| \frac{\partial^N u_k}{\partial t^N}(x, t) \right| \leq \frac{c}{t^{2\gamma+d+N}}$ ;  $k \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ .

ii) For  $\|x\|$  large we have

$$u_k(x, t) \leq \frac{c}{(t^2 + \|x\|^2)^{\gamma+d/2}} \quad \text{and} \quad \left| \frac{\partial u_k}{\partial x_i}(x, t) \right| \leq \frac{c}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}}.$$

*Proof.* i) If the generalized Poisson kernel  $P_t$  is a positive radial function, then from Proposition 2.5 i) we obtain  $u_k(x, t) \geq 0$ .



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On the other hand from Proposition 2.5 iii) we have

$$\left| \frac{\partial^N u_k}{\partial t^N}(x, t) \right| \leq \|f\|_{L_k^1} \left\| \frac{\partial^N P_t}{\partial t^N} \right\|_{L_k^\infty}.$$

Then we obtain the result from the fact that

$$\left\| \frac{\partial^N P_t}{\partial t^N} \right\|_{L_k^\infty} \leq \frac{c}{t^{2\gamma+d+N}}.$$

ii) From Proposition 2.5 i) we can write

$$\tau_x P_t(-y) = a_k \int_{\mathbb{R}^d} \frac{t d\Gamma_x(\xi)}{[t^2 + \|x\|^2 + \|y\|^2 - 2\langle y, \xi \rangle]^{\gamma+(d+1)/2}}; \quad x, y \in \mathbb{R},$$

where  $a_k$  is the constant given by (3.2).

Since  $f \in \mathcal{D}(\mathbb{R}^d)$ , there exists  $a > 0$ , such that  $\text{supp}(f) \subset B_d(o, a)$ . Then

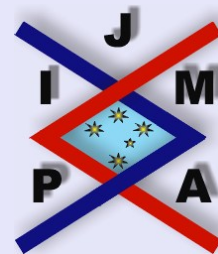
$$u_k(x, t) = a_k \int_{B_d(o, a)} \int_{\mathcal{A}_{x, y}} \frac{t f(y) d\Gamma_x(\xi) d\mu_k(y)}{[t^2 + \|x\|^2 + \|y\|^2 - 2\langle y, \xi \rangle]^{\gamma+(d+1)/2}}.$$

It is easily verified for  $\|x\|$  large and  $y \in B_d(o, a)$  that

$$\frac{1}{[t^2 + \|x\|^2 + \|y\|^2 - 2\langle y, \xi \rangle]^{\gamma+(d+1)/2}} \leq \frac{c}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}}.$$

Therefore and using the fact that  $t \leq (t^2 + \|x\|^2)^{1/2}$ , we obtain

$$u_k(x, t) \leq \frac{c t}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}} \leq \frac{c}{(t^2 + \|x\|^2)^{\gamma+d/2}}.$$



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Thus the first inequality is proven.

From (2.6) we can write

$$u_k(x, t) = a_k \int_{B_d(o, a)} \int_{\mathcal{A}_{x, y}} \frac{t f(-y) d\Gamma_y(\xi) d\mu_k(y)}{[t^2 + \|x\|^2 + \|y\|^2 + 2\langle x, \xi \rangle]^{\gamma+(d+1)/2}}.$$

By derivation under the integral sign we obtain

$$\frac{\partial u_k}{\partial x_i}(x, t) = a_k \int_{B_d(o, a)} \int_{\mathcal{A}_{x, y}} \frac{-t(2x_i + \xi_i) f(-y) d\Gamma_y(\xi) d\mu_k(y)}{[t^2 + \|x\|^2 + \|y\|^2 + 2\langle x, \xi \rangle]^{\gamma+(d+3)/2}}.$$

But for  $\|x\|$  large and  $y \in B_d(o, a)$  we have

$$\frac{t|2x_i + \xi_i|}{[t^2 + \|x\|^2 + \|y\|^2 + 2\langle x, \xi \rangle]^{\gamma+(d+3)/2}} \leq \frac{t(2|x_i| + |\xi_i|)}{(t^2 + \|x\|^2)^{\gamma+(d+3)/2}}.$$

Using the fact that  $t(2|x_i| + |\xi_i|) \leq (1 + |\xi_i|)(t^2 + \|x\|^2)$  when  $\|x\|$  large, we obtain

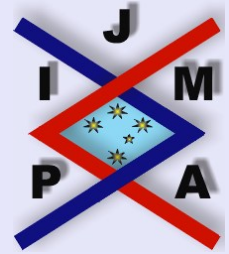
$$\left| \frac{\partial u_k}{\partial x_i}(x, t) \right| \leq \frac{c}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}},$$

which proves the second inequality. □

**Lemma 3.3.** *Let  $f \in \mathcal{D}(\mathbb{R}^d)$  be a positive function and  $p \in ]1, \infty[$ .*

$$i) \lim_{N \rightarrow \infty} \int_{B_d(o, N)} \int_0^N \frac{\partial^2 u_k^p}{\partial t^2}(x, t) t dt d\mu_k(x) = \int_{\mathbb{R}^d} f^p(x) d\mu_k(x).$$

$$ii) \lim_{N \rightarrow \infty} \int_0^N \int_{B_d(o, N)} L_k u_k^p(\cdot, t)(x) d\mu_k(x) t dt = 0,$$



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where  $L_k$  is the singular elliptic operator given by (2.4).

*Proof.* i) Integrating by parts, we obtain

$$\begin{aligned} & \int_{B_d(o,N)} \int_0^N \frac{\partial^2 u_k^p}{\partial t^2}(x,t) t dt d\mu_k(x) \\ &= \int_{B_d(o,N)} f^p(x) d\mu_k(x) - \int_{B_d(o,N)} u_k^p(x,N) d\mu_k(x) \\ & \quad + pN \int_{B_d(o,N)} u_k^{p-1}(x,N) \frac{\partial u_k}{\partial t}(x,N) d\mu_k(x). \end{aligned}$$

From Lemma 3.2 i), we easily get

$$\int_{B_d(o,N)} u_k^p(x,N) d\mu_k(x) \leq c N^{-(p-1)(2\gamma+d)},$$

and

$$N \int_{B_d(o,N)} u_k^{p-1}(x,N) \frac{\partial u_k}{\partial t}(x,N) d\mu_k(x) \leq c N^{-(p-1)(2\gamma+d)},$$

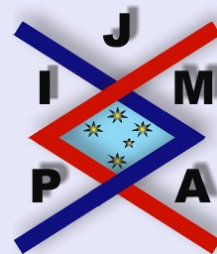
which gives i).

ii) We have

$$\int_0^N \int_{B_d(o,N)} L_k u_k^p(\cdot, t)(x) d\mu_k(x) t dt = \sum_{i=1}^d I_{i,N},$$

where

$$I_{i,N} = \int_0^N \int_{B_d(o,N)} \frac{\partial}{\partial x_i} \left( w_k(x) \frac{\partial u_k^p}{\partial x_i}(x,t) \right) dx t dt, \quad i = 1, \dots, d.$$



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Let us study  $I_{1,N}$ :

$$I_{1,N} = p \int_0^N \int_{B_{d-1}(o,N)} w_k(x^{(N)}) \left[ u_k^{p-1}(x^{(N)}, t) \frac{\partial u_k}{\partial x_1}(x^{(N)}, t) - u_k^{p-1}(-x^{(N)}, t) \frac{\partial u_k}{\partial x_1}(-x^{(N)}, t) \right] dx_2 \dots dx_d t dt,$$

where  $x^{(N)} = \left( \sqrt{N^2 - \sum_{i=2}^d x_i^2}, x_2, \dots, x_d \right)$ .

Then, by using Lemma 3.2 ii) and the fact that  $w_k(x^{(N)}) \leq 2^\gamma N^{2\gamma}$  we obtain for  $N$  large,

$$\begin{aligned} I_{1,N} &\leq c N^{2\gamma} \int_0^N \int_{B_{d-1}(o,N)} \frac{dx_2 \dots dx_d t dt}{(t^2 + N^2)^{(\gamma+d/2)p+1/2}} \\ &\leq c N^{-p(2\gamma+d)+2\gamma-1} \int_0^N \int_{B_{d-1}(o,N)} dx_2 \dots dx_d t dt \\ &\leq c N^{-(p-1)(2\gamma+d)-(d-1)/2}. \end{aligned}$$

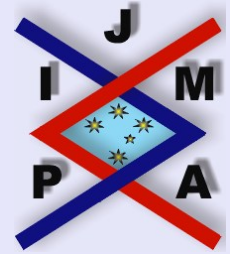
The same result holds for  $I_{i,N}$ ,  $i = 2, \dots, d$ , which proves ii). □

**Lemma 3.4.** Let  $f \in \mathcal{D}(\mathbb{R}^d)$  be a positive function. Define the maximal function  $\mathcal{M}_k(f)$ , by

$$(3.3) \quad \mathcal{M}_k(f)(x) := \sup_{t>0} (u_k(x, t)), \quad x \in \mathbb{R}^d.$$

Then for  $p \in ]1, \infty[$ , there exists a constant  $C_p > 0$  such that, for  $f \in L_k^p(\mathbb{R}^d)$ ,

$$\|\mathcal{M}_k(f)\|_{L_k^p} \leq C_p \|f\|_{L_k^p},$$



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moreover the operator  $\mathcal{M}_k$  is of weak type  $(1, 1)$ .

*Proof.* From (3.1) it follows that

$$u_k(x, t) = \frac{t}{8\sqrt{\pi}} \int_0^\infty W_s *_k f(x) e^{-t^2/4s} s^{-3/2} ds,$$

which implies, as in [18, p. 49] that

$$\mathcal{M}_k(f)(x) \leq c \sup_{y>0} \left( \frac{1}{y} \int_0^y Q_s f(x) ds \right), \quad x \in \mathbb{R}^d,$$

where  $Q_s f(x) = W_s *_k f(x)$ , which is a semigroup of operators on  $L_k^p(\mathbb{R}^d)$ . Hence using the Hopf-Dunford-Schwartz ergodic theorem as in [18, p. 48], we get the boundedness of  $\mathcal{M}_k$  on  $L_k^p(\mathbb{R}^d)$  for  $p \in ]1, \infty]$  and weak type  $(1, 1)$ .  $\square$

*Proof of Theorem 3.1.* Let  $f \in \mathcal{D}(\mathbb{R}^d)$  be a positive function. From Lemma 3.2 i) the generalized Poisson integral  $u_k$  of  $f$  is positive.

*First step:* Estimate of the quantity  $|\frac{\partial}{\partial t} u_k(x, t)|^2 + |\nabla_x u_k(x, t)|^2$ .

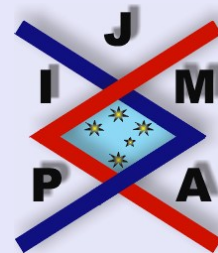
Let  $\mathcal{H}_k$  be the operator:

$$\mathcal{H}_k := L_k + \frac{\partial^2}{\partial t^2},$$

where  $L_k$  is the singular elliptic operator given by (2.3).

Using the fact that

$$\Delta_k u_k(\cdot, t)(x) + \frac{\partial^2}{\partial t^2} u_k(x, t) = 0,$$



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we obtain for  $p \in ]1, \infty[$ ,

$$\mathcal{H}_k u_k^p(x, t) = p(p-1)u_k^{p-2}(x, t) \left[ \left| \frac{\partial}{\partial t} u_k(x, t) \right|^2 + |\nabla_x u_k(x, t)|^2 \right] + p \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \frac{U_\alpha(x, t)}{\langle \alpha, x \rangle^2},$$

where

$$U_\alpha(x, t) := 2u_k^{p-1}(x, t) [u_k(x, t) - u_k(\sigma_\alpha x, t)], \quad \alpha \in \mathbb{R}_+.$$

Let  $A, B \geq 0$ , then the inequality

$$2A^{p-1}(A - B) \geq (A^{p-1} + B^{p-1})(A - B)$$

is equivalent to

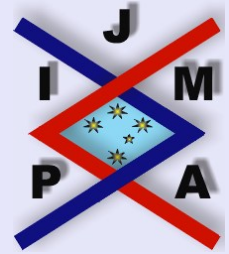
$$(A^{p-1} - B^{p-1})(A - B) \geq 0,$$

which holds if  $A \geq B$  or  $A < B$ . Thus we deduce that

$$U_\alpha(x, t) \geq [u_k^{p-1}(x, t) + u_k^{p-1}(\sigma_\alpha x, t)] [u_k(x, t) - u_k(\sigma_\alpha x, t)],$$

and therefore we get

$$(3.4) \quad \left| \frac{\partial}{\partial t} u_k(x, t) \right|^2 + |\nabla_x u_k(x, t)|^2 \leq \frac{1}{p(p-1)} u_k^{2-p}(x, t) [v_k(x, t) + \mathcal{H}_k u_k^p(x, t)],$$



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where

$$v_k(x, t) = p \sum_{\alpha \in R_+} \frac{k(\alpha)}{\langle \alpha, x \rangle^2} [u_k^{p-1}(\sigma_\alpha x, t) + u_k^{p-1}(x, t)] [u_k(\sigma_\alpha x, t) - u_k(x, t)].$$

*Second step:* The inequality  $\|g(f)\|_{L_k^p} \leq A_p \|f\|_{L_k^p}$ , for  $p \in ]1, 2[$ .

From (3.4), we have

$$\begin{aligned} [g(f)(x)]^2 &\leq \frac{1}{p(p-1)} \int_0^\infty u_k^{2-p}(x, t) [v_k(x, t) + \mathcal{H}_k u_k^p(x, t)] t dt \\ &\leq \frac{1}{p(p-1)} \mathcal{I}_k(f)(x) [\mathcal{M}_k(f)(x)]^{2-p}, \quad x \in \mathbb{R}^d, \end{aligned}$$

where

$$\mathcal{I}_k(f)(x) := \int_0^\infty [v_k(x, t) + \mathcal{H}_k u_k^p(x, t)] t dt,$$

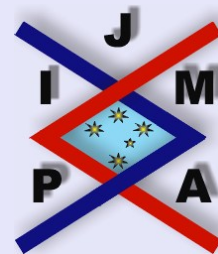
and  $\mathcal{M}_k(f)$  the maximal function given by (3.3).

Thus it is proven that

$$\|g(f)\|_{L_k^p}^p \leq \left( \frac{1}{p(p-1)} \right)^{\frac{p}{2}} \int_{\mathbb{R}^d} [\mathcal{I}_k(f)(x)]^{p/2} [\mathcal{M}_k(f)(x)]^{(2-p)p/2} d\mu_k(x).$$

By applying Hölder's inequality, we obtain

$$(3.5) \quad \|g(f)\|_{L_k^p}^p \leq \left( \frac{1}{p(p-1)} \right)^{\frac{p}{2}} \|\mathcal{I}_k(f)\|_{L_k^1}^{p/2} \|\mathcal{M}_k(f)\|_{L_k^p}^{(2-p)p/2}.$$



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Since  $v_k(x, t) + \mathcal{H}_k u_k^p(x, t) \geq 0$ , we can apply Fubini-Tonnelli's Theorem to obtain

$$\|\mathcal{I}_k(f)\|_{L_k^1} = \lim_{N \rightarrow \infty} \int_0^N \int_{B_d(o, N)} [v_k(x, t) + \mathcal{H}_k u_k^p(x, t)] d\mu_k(x) t dt.$$

Putting  $y = \sigma_\alpha x$  and using the fact that  $\sigma_\alpha^2 = id$ ;  $\langle \sigma_\alpha y, \alpha \rangle = -\langle y, \alpha \rangle$ , then as in the argument of [16, p. 390] we obtain

$$\int_{B_d(o, N)} v_k(x, t) d\mu_k(x) = - \int_{B_d(o, N)} v_k(y, t) d\mu_k(y).$$

Thus

$$\int_{B_d(o, N)} v_k(x, t) d\mu_k(x) = 0.$$

Hence from Lemma 3.3, we deduce that

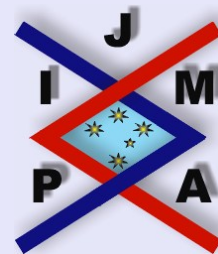
$$(3.6) \quad \|\mathcal{I}_\alpha(f)\|_{L_k^1} = \lim_{N \rightarrow \infty} \int_{B_d(o, N)} \int_0^N \mathcal{H}_k u_k^p(x, t) t dt d\mu_k(x) = \|f\|_{L_k^p}^p.$$

On the other hand from Lemma 3.4 we have

$$(3.7) \quad \|\mathcal{M}_k(f)\|_{L_k^p} \leq C_p \|f\|_{L_k^p}.$$

Finally, from (3.5), (3.6) and (3.7), we obtain

$$\|g(f)\|_{L_k^p} \leq A_p \|f\|_{L_k^p}, \quad A_p = \left( \frac{1}{p(p-1)} \right)^{\frac{1}{2}} C_p^{(2-p)/2}.$$



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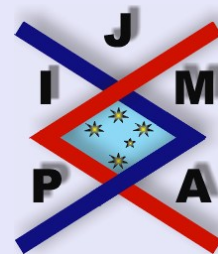


Since the operator  $g$  is sub-linear, we obtain the inequality for  $f \in \mathcal{D}(\mathbb{R}^d)$ . And by an easy limiting argument one shows that the result is also true for any  $f \in L_k^p(\mathbb{R}^d)$ ,  $p \in ]1, 2[$ .

For the case  $p = 2$ , using (3.4) and (3.6) we get

$$\|g(f)\|_{L_k^2}^2 \leq \frac{1}{2} \int_{\mathbb{R}^d} \int_0^\infty [v_k(x, t) + \mathcal{H}_k u_k^2(x, t)] t dt d\mu_k(x) = \frac{1}{2} \|f\|_{L_k^2}^2,$$

which completes the proof of the theorem. □




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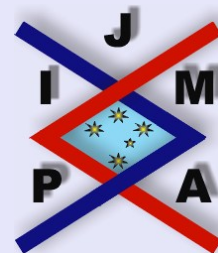
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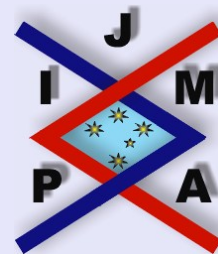
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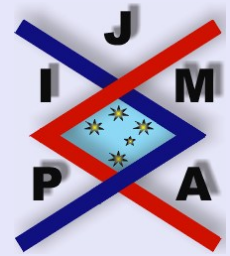
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